A new approach for reconstruction of γ-multiplicity distribution from measured γ-fold distribution

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Angular momentum distribution in fusion reactions is estimated indirectly from the γ-multiplicity distribution i.e. the distribution of the number of γ-quanta emitted from the compound nucleus after fusion. Since 100% detection efficiency for the γ-rays can not be achieved experimentally, one can record only a fraction of the total number of γ-rays emitted. The γ-fold distribution, measured experimentally, is unfolded to obtain some information about the γ-multiplicity distribution. The γ-fold probabilities $P(p)$ can be expanded in terms of the factorial moments of the γ-multiplicity distribution as given below [1]

$$P(p) = \sum_{m=0}^{\infty} \left[\left\langle \binom{M}{m} m! \right\rangle \right] A_{\rm pm}(\Omega) \qquad (1)$$

where $\mathbf{A}$ is a $(N+1)\times\infty$ dimensional matrix and its elements are given by

$$A_{pm} = \frac{(-1)^m}{m!} \binom{N}{p} \sum_{l=0}^{p} (-1)^{p-l} \binom{p}{l} (N-l)^m \Omega^m.$$

Solving for the lowest factorial moments (up to N) by inverting the left $(N+1)\times(N+1)$ block of $\mathbf{A}$, we get

$$\left\langle \binom{M}{m} m! \right\rangle = \sum_{p=0}^{N} A^{-1}_{\rm mp} P(p). \qquad (2)$$

The matrix $\mathbf{A^{-1}}$ is the response matrix of the detector array which is dependent on the number of detectors (N) and the efficiency of each detector (Ω) in the array. The factorial moments, given by Eq. 2, can be transformed to raw moments of the γ-multiplicity distribution. Only a small number of moments (typically few tens) can be extracted experimentally due to practical limitations in the experimental set up (specifically, no. of detectors).

To obtain the γ-multiplicity distribution itself, one usually ignores the fact that the distribution is discrete and assumes a simple a priori continuous shape of the distribution [2]. Here we consider the reconstruction of a discrete probability distribution using a finite number of its moments, for cases in which *the number of known moments equals or exceeds the extent of the distribution.*

Let $x = \{0, 1, \ldots, i, \ldots, K\}$ be a countable discrete positive random variable with probability mass function (PMF) $F = \{f_0, f_1, \ldots, f_i, \ldots, f_K\}$. Then the n-th raw moment of F is

$$\mu_n = \mathbb{E}(x^n) = \sum_{i=0}^{K} i^n f_i. \qquad (3)$$

Eq. 3, written explicitly for $n = 0$ to K, yields a set of $(K+1)$ linear equations

$$\begin{aligned} \mu_0 &= 0^0 f_0 + \ldots + i^0 f_i + \ldots + K^0 f_K \\ \mu_1 &= 0^1 f_0 + \ldots + i^1 f_i + \ldots + K^1 f_K \\ &\ldots \\ \mu_n &= 0^n f_0 + \ldots + i^n f_i + \ldots + K^n f_K \\ &\ldots \\ \mu_K &= 0^K f_0 + \ldots + i^K f_i + \ldots + K^K f_K. \end{aligned} \qquad (4)$$

Eq. 4 can be rewritten in matrix notation as

$$\begin{pmatrix} \mu_0 \\ \mu_1 \\ \ldots \\ \mu_n \\ \ldots \\ \mu_K \end{pmatrix} = \begin{pmatrix} 0^0 & \ldots & i^0 & \ldots & K^0 \\ 0^1 & \ldots & i^1 & \ldots & K^1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0^n & \ldots & i^n & \ldots & K^n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0^K & \ldots & i^K & \ldots & K^K \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ \ldots \\ f_n \\ \ldots \\ f_K \end{pmatrix} \qquad (5)$$

or, in short, as

$$\mathbf{TF} = \boldsymbol{\mu} \qquad (6)$$

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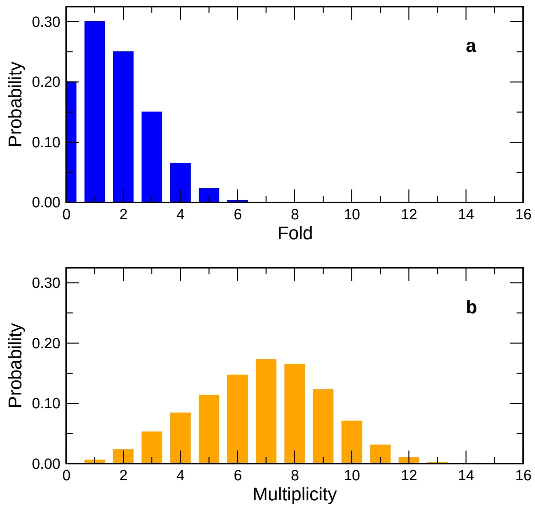

FIG. 1: (a) Experimental γ-fold distribution for the reaction ^{16}O+^{115}In, measured at 60 MeV beam energy using an array of 15 BGO detectors [3] and (b) the corresponding γ-multiplicity distribution, reconstructed using the proposed method.

where $\mathbf{T}$ is the $(K+1)\times(K+1)$ dimensional matrix in Eq. 5. Thus, one can determine $\mathbf{F}$ *exactly*, if $\mathbf{T^{-1}}$ exists,

$$\mathbf{F} = \mathbf{T^{-1}}\boldsymbol{\mu}. \tag{7}$$

One can notice here that the matrix $\mathbf{T}$ is of a very simple form and each element of the matrix is given by (column no. - 1)$^{\text{(row no. - 1)}}$. This follows from the fact that we are considering a discrete distribution. Thus, the problem of reconstructing the PMF F from a finite number of its moments boils down to inverting the matrix $\mathbf{T}$.

We have applied the proposed method to reconstruct the γ-multiplicity distribution in the fusion reaction ^{16}O+^{115}I, reported by Nayak *et al.* [3]. The γ-fold distribution for this reaction at 60 MeV beam energy is reproduced in the top panel of Fig. 1. We have calculated raw moments of the γ-multiplicity distribution from the γ-fold probabilities. We have next reconstructed the γ-multiplicity distribution using Eq. 7, which is shown in the bottom panel of Fig. 1.

It has already been mentioned that the method can be applied in cases when the extent of the γ-multiplicity distribution is less than or equal to the number of available moments. Thus, we require an array of sufficiently large number of detectors for universal applicability of the method in γ-multiplicity measurements. Besides experimental limitations (resources, available space, mechanical support structure and electronic signal processing), data from a large array pose computational challenges. The dimensions of the matrices $\mathbf{A}$ and $\mathbf{T}$ increase linearly with increasing size of the array. One must be very careful while inverting these two matrices. A good programming practice is to check the unitarity of the product of each matrix with its inverse i.e.

$$\mathbf{A^{-1}A} = \mathbf{I} = \mathbf{T^{-1}T} \tag{8}$$

where $\mathbf{I}$ is the unit matrix of same dimension.

We have found that beyond a certain dimension of $\mathbf{A}$ and $\mathbf{T}$ (dependent on the machine used for computation), the default machine precision is not adequate. Upon noticing deviation from the condition stated in Eq. 8, we have switched from default machine precision to multiple precision calculation [4].

It is to be noted here that no additional uncertainties are added to the multiplicity distribution in the process of its reconstruction from the moments, as the proposed method is an *exact* one. The errors in the calculation of the moments, primarily caused by the uncertainties in determining the absolute efficiency of the detectors, only contribute to the uncertainties in the reconstructed γ-multiplicity distribution. Encouraged by the first results, presented here, we may look forward to reconstruction of γ-multiplicity distribution by the proposed method from γ-fold data obtained using larger arrays.

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