Quantum resonances built on classical nonlinear resonances

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Physics of rotating and vibrating nuclei is central to our understanding of their structural and reactive aspects [1]. An interplay of single-particle motion with the shape degrees of freedom gives an inertia to a nucleus. The equilibrium shapes are generally integrable for which the energy levels are easily found. However, as soon as the system begins to rotate or vibrate, we have a particle in a deforming box, confined but still not stationary. Classically, there will be nonlinear resonances. Due to these, in certain regions in phase space, particles are trapped. The aim of this paper is to understand the quantum analogue of this phenomenon. For simplicity, as a representative of a regular system, we consider a particle in a two-dimensional rectangular box (simple integrable) rotating at a constant angular velocity $\omega > 0$ (already non-integrable). We deal with the case when the rotation is slow i.e. $\omega \ll 1$. The Hamiltonian of the system in the rotating coordinate frame is $H = \frac{1}{2}\left(p_1^2 + p_2^2\right) + \omega\left(p_1 q_2 - p_2 q_1\right) + U\left(q_1, q_2\right)$ where p_i, q_i are the canonically conjugate momenta and coordinates respectively, and $U(q_1, q_2)$ is the hard wall potential of the box. We make a change of coordinates to the action angle variables I_i, ϕ_i such that for $0 < \phi_i \le \pi$, $q_i = -d_i + \frac{2d_i\phi_i}{\pi}$, and, for $\pi < \phi_i \le 2\pi$, $3d_i - \frac{2d_i\phi_i}{\pi}$; $p_i = \frac{\pi I_i}{2d_i}\mathrm{sgn}\left(\sin\phi_i\right)$. Here $2d_i$ is the side of the rectangular box. We assume that the sides of the box are moving slowly with time and hence the length of walls are time dependent, $d_i = d_i\left(\epsilon t\right)$ where $\epsilon \ll 1$. Since p_i, q_i are periodic in ϕ_i, therefore the variables p_i, q_i can be written as Fourier series of I_i, ϕ_i. The new Hamiltonian is [2]

$$H = h_0 + f(\phi_1, \phi_2) - \frac{8\omega}{\pi^2}$$
$$\left[\sum_{k1,k2}{}' \left(\frac{I_1 d_2^2 k_1 - I_2 d_2^2 k_2}{d_1 d_2 k_1^2 k_2^2}\right) \sin\varphi_1 \right.$$
$$\left. + \sum_{k1,k2}{}' \left(\frac{I_1 d_2^2 k_1 + I_2 d_2^2 k_2}{d_1 d_2 k_1^2 k_2^2}\right) \sin\varphi_2 \right]$$

where $h_0 = \frac{\pi^2}{8}\left(\frac{I_1^2}{d_1^2} + \frac{I_2^2}{d_2^2}\right)$, $\varphi_{1,2} = k_1\phi_1 \pm k_2\phi_2$, and, $f(\phi_1, \phi_2) = \epsilon(E_1(\phi_1) + E_2(\phi_2))$. For the unperturbed hamiltonian, $\omega_1 = \frac{\pi^2 I_1}{4d_1^2}$, $\omega_2 = \frac{\pi^2 I_2}{4d_2^2}$. Therefore the nonlinear resonance occurs when $m\omega_1 - n\omega_2 = 0$ and $\frac{mI_1}{d_1^2} - \frac{nI_2}{d_2^2} = 0$. Near the (m,n) resonance, we make a change of coordinates using the generating function, $W = \left(m\phi_1 - n\phi_2\right)R - \left(l_1\phi_1 - l_2\phi_2\right)J$. Here R and J are the new action coordinates and ϕ, ψ are the new conjugate angle coordinates. The constants m, n, l_1, l_2 satisfies $ml_2 - nl_1 = 1$. From this transformation we get $I_1 = mR - l_1 J$, $I_2 = -nR + l_2 J$, $\phi = m\phi_1 - n\phi_2, \psi = -l_1\phi_1 + l_2\phi_2$ The resonance condition in new coordinates becomes $R = R_{res}\left(J, \epsilon t\right) = \frac{nl_2 d_1^2 + ml_1 d_2^2}{m^2 d_2^2 + n^2 d_2^2} J$. In these new variables Hamiltonian will be $H = H_0\left(R, J, \epsilon t\right) + \omega H_1\left(R, J, \phi, \psi, \epsilon t\right) + \frac{\partial W}{\partial t}$. The analytical form for H_0 and H_1 are rather complicated. Near an (m,n) resonance, $\dot{\phi} \approx 0$. Therefore ψ is the faster variable and we will

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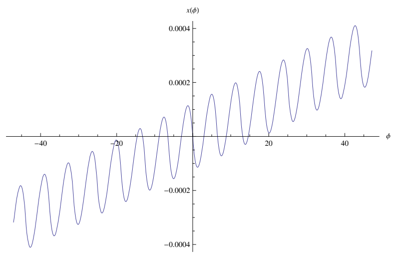

FIG. 1: The resonance Hamiltonian for the parameters near (1,3) resonance after 10 s (arb. units): $\epsilon = 4 \times 10^{-4}, \omega = 1.5 \times 10^{-3}, d_1 = 1 + 0.1 \cos(\epsilon t), d_2 = 2\left(1 + 0.15 \cos\left(\epsilon t + \frac{\pi}{3}\right)\right)$, starting at $t = 0$.

average over it:

$$\overline{H_0} = H_0$$

$$\overline{H_1} = -\frac{8\omega}{\pi^2} \sum \left[\frac{R(m^2 d_2^2 r - n^2 d_1^2 r)}{r^4 d_1 d_2 m^2 n^2} + \frac{J(r l_2 d_1^2 n - r l_1 d_2^2 m)}{r^4 d_1 d_2 m^2 n^2} \right] \sin(r\phi)$$

$$\overline{H} = \overline{H_0} + \overline{H_1} + \overline{\frac{\partial W}{\partial t}}. \tag{1}$$

Near a resonance the Hamiltonian can be approximated to order ϵ, writing $R - R_{res} = P$ we get $H = \Lambda(J, \epsilon t) + F_0 + O(\epsilon)$ and $F_0 = \frac{1}{2} g P^2 + b\chi(\phi) + a\phi$. On the resonance, Λ is H_0, and $g = \left(\frac{\partial^2 H_0}{\partial R^2}\right)_{R=R_{res}} = \frac{\pi^2}{4}\left(\frac{k_1^2}{d_1^2} + \frac{k_2^2}{d_2^2}\right)$, $b = -2\frac{\omega d_1 d_2 J}{k_1 k_2 \left(k_1^2 d_2^2 + k_2^2 d_1^2\right)}$, $a = \epsilon \frac{\partial R_{res}}{\partial \epsilon t}$, and

$$\chi(\phi) = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^3} \sin n\phi = \begin{cases} \phi - \frac{\phi^2}{\pi} & , 0 < \phi < \pi \\ \phi + \frac{\phi^2}{\pi} & , -\pi < \phi \leq 0. \end{cases}$$

The dynamics near resonance depends on the properties of the resonance Hamiltonian F_0. For fixed J and t, F_0 is a Wannier-Stark ladder (Fig. 1), leading to resonance energy states with finite lifetimes.

As time evolves the depth of the valleys in the resonance hamiltonian change periodically and new states are born and disappear with time. For some specific forms of d_1 and d_2, the minima disappear for certain time in the resonance hamiltonian which means that no bound states are possible.

To find these resonance states we use the method given by Gluck et al. [3] where we define the Wannier-Bloch state satisfying $\hat{W}^{(k)} \chi_{l,k}(x) = U \chi_{l,k}(x)$, where

$$U = \exp\left[-\iota \frac{T_B}{\hbar}\left(E_l(k) - \frac{\Gamma_l(k)}{2}\right)\right]$$

$$\hat{W}^{(k)} = e^{-x} \widehat{\exp}\left(-\frac{\iota}{\hbar} \int_0^{T_B} \widetilde{F_0}(t')\, dt'\right)$$

$$\widetilde{F_0}(t) = \frac{1}{2} g (P - at)^2 + b\chi(\phi) \tag{2}$$

and $T_B = \frac{\hbar}{a}$ is the Bloch Period. Numerically we calculate the resonance energy states by diagonalizing the nonunitary matrix $\widehat{W^{(k)}}(n, n')$ formed in the periodic function $|n >= \frac{1}{\sqrt{2\pi}} \exp(\iota n x)$ basis. The ten most stable states and their decay coefficients are given below (taking $\hbar = 0.5$):

Energy level ($\times 10^{-7}$)	Decay Coefficient ($\frac{\Gamma}{2} \times 10^{-6}$)
92.137	2.913
−146.126	2.955
−4.444	3.003
−118.605	3.044
66.967	3.044
20.750	3.071
−169.798	3.100
−28.527	3.127
118.384	3.146
−53.449	3.153

Therefore near a resonance we get these energy levels with finite lifetime. Thus, there might be certain regions created by nonlinear resonances, classically inaccessible and disjoint, but quantum mechanically supporting states with finite lifetime in bound systems !

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