# New derivation of relativistic dissipative fluid dynamics

Amaresh Jaiswal,* Rajeev S. Bhalerao, and Subrata Pal
*Tata Institute of Fundamental Research, Mumbai - 400005, INDIA*

## Introduction

Relativistic dissipative hydrodynamics has been quite successful in explaining the spectra and azimuthal anisotropy of particles produced in heavy-ion collisions at the RHIC and recently at the LHC [1,2]. The first-order dissipative fluid dynamics or the relativistic Navier-Stokes (NS) theory involves parabolic differential equations and suffers from acausality and instability. The second-order or Israel-Stewart (IS) theory [3] with its hyperbolic equations restores causality but may not guarantee stability. The correct formulation of relativistic viscous fluid dynamics is far from settled and is under intense investigation.

It is important to note that all formulations of the second-order viscous hydrodynamics that employ the Boltzmann equation (BE) make a strict assumption of locality in the configuration space for the collision term [3]. In other words, the collisions that increase or decrease the number of particles with a given momentum $p$, in a small space-time volume, are assumed to occur at the same point $x^\mu$. This makes the collision integral a purely local functional of the single-particle phase-space distribution function $f(x,p)$ independent of the derivatives $\partial^\mu f$. Although $f(x,p)$ may not vary significantly over the length scale of a single collision event, its variation over the length scales extending over several mean interparticle spacings may not be negligible. Including the gradients of $f(x,p)$ in the collision term gives rise to different evolution equations for the dissipative quantities.

We provide a new formulation of the dissipative hydrodynamic equations within kinetic theory by using a nonlocal collision term in the Boltzmann equation: $p^\mu\partial_\mu f = C[f]$.

*Electronic address: amaresh@tifr.res.in

## Non-locality in collision term

For two-body elastic collisions, the collision term for classical Boltzmann gas is

$$C[f] = \frac{1}{2}\int dp' dk\, dk'\, W_{pp'\to kk'}(f_k f_{k'} - f_p f_{p'}), \tag{1}$$

where $W_{pp'\to kk'}$ is the collisional transition rate, $f_p \equiv f(x,p)$ and $dp = d\mathbf{p}/[(2\pi)^3 E_\mathbf{p}]$. The first and second terms in Eq. (1) refer to the processes $kk' \to pp'$ and $pp' \to kk'$, respectively. These processes are traditionally assumed to occur at the same space-time point $x^\mu$ with an underlying assumption that $f(x,p)$ is constant not only over a region characterizing a single collision, but also over an infinitesimal fluid element of size $dR$, large compared to the interparticle separation. We emphasize that the space-time points at which the above two processes occur may be separated by a small interval $\xi^\mu$ within $d^4R$. With this realistic viewpoint, the second term in Eq. (1) involves $f(x-\xi,p)f(x-\xi,p')\tilde{f}(x-\xi,k)\tilde{f}(x-\xi,k')$, which on Taylor expansion up to second order in $\xi^\mu$ , results in modified BE

$$p^\mu\partial_\mu f = C[f] + \partial_\mu(A^\mu f) + \partial_\mu\partial_\nu(B^{\mu\nu} f), \tag{2}$$

The momentum dependence of coefficients,

$$B^{\mu\nu} = -\frac{1}{4}\int dp' dk\; dk'\; \xi^\mu\xi^\nu W_{pp'\to kk'} f_{p'},$$
$$A^\mu = \frac{1}{2}\int dp' dk\; dk'\; \xi^\mu W_{pp'\to kk'} f_{p'}, \tag{3}$$

can be made explicit by expressing them in terms of the available tensors $p^\mu$ and the metric $g^{\mu\nu} \equiv \mathrm{diag}(1,-1,-1,-1)$ as $A^\mu = ap^\mu$ and $B^{\mu\nu} = b_1 g^{\mu\nu} + b_2 p^\mu p^\nu$. The scalar coefficients $a$, $b_1$ and $b_2$ are functions of $x^\mu$.

The conserved particle current, $N^\mu = \int dp\; p^\mu f$, and the energy-momentum tensor, $T^{\mu\nu} = \int dp\; p^\mu p^\nu f$ has the standard tensor decomposition [1,3]:

$$
\begin{aligned}
N^{\mu} &= nu^{\mu} + n^{\mu}, \\
T^{\mu\nu} &= \epsilon u^{\mu}u^{\nu} - (P+\Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}, \quad (4)
\end{aligned}
$$

Conservation of current and energy-momentum in microscopic interactions leads to three constraint equations for the coefficients $(a, b_1, b_2)$ [4], namely $\partial_\mu a = 0$,

$$
\begin{aligned}
\partial^2\left(b_1 a_{00}\right) + \partial_\mu\partial_\nu\left(b_2 I^{\mu\nu}\right) &= 0, \\
u_\alpha\partial_\mu\partial_\nu\left(b_2 I^{\mu\nu\alpha}\right) + u_\alpha\partial^2\left(b_1 n u^\alpha\right) &= 0. \quad (5)
\end{aligned}
$$

In order to obtain the second-order evolution equations for viscous quantities (only shear is considered here), we take the second moment of the modified Boltzmann equation (2)

$$
\int dp\ p^\alpha p^\beta p^\gamma \partial_\gamma f = \int dp\ p^\alpha p^\beta \big[C[f] + p^\gamma\partial_\gamma(af) + \partial^2(b_1 f_0) + (p\cdot\partial)^2(b_2 f_0)\big]. \quad (6)
$$

Using Grad's 14-moment approximation for the single particle distribution function, $f = f_0\left(1 + \lambda_\pi \pi_{\alpha\beta}p^\alpha p^\beta\right)$, we finally obtain the following evolution equation for shear tensor

$$
\begin{aligned}
\pi^{\mu\nu} = {} & \tilde{a}\pi^{\mu\nu}_{\rm NS} - \beta_{\dot\pi}\tau_\pi\dot\pi^{\langle\mu\nu\rangle} + \lambda_{\pi\pi}\pi_\rho^{\langle\mu}\omega^{\nu\rangle\rho} \\
& - \tau_{\pi\pi}\pi_\rho^{\langle\mu}\sigma^{\nu\rangle\rho} - \delta_{\pi\pi}\pi^{\mu\nu}\theta + \Lambda_{\pi\dot u}\dot u^{\langle\mu}\dot u^{\nu\rangle} \\
& + \Lambda_{\pi\omega}\omega_\rho^{\langle\mu}\omega^{\nu\rangle\rho} + \chi_1\dot b_2\pi^{\mu\nu} \\
& + \chi_2\dot u^{\langle\mu}\nabla^{\nu\rangle}b_2 + \chi_3\nabla^{\langle\mu}\nabla^{\nu\rangle}b_2, \quad (7)
\end{aligned}
$$

in the usual notations [4].

## Results and discussions

The coupled differential equations (5), (7) and $\partial_\mu T^{\mu\nu} = 0$ are solved for the evolution of a massless Boltzmann gas, at vanishing net baryon density in the 1-D Bjorken model.

Figure 1(a) shows the evolution of several quantities for a particular choice of initial conditions. $T$ decreases monotonically to the crossover temperature $T_c \simeq 170$ MeV at time $\tau \simeq 10$ fm/c. Parameters $b_1$, $b_2$ vary smoothly to zero at large times indicating reduced but still significant presence of nonlocal effects at late times. This is also evident in Fig. 1(b) where the pressure anisotropy $P_L/P_T = (P_0 - \pi)/(P_0 + \pi/2)$ shows marked deviation from IS. Although the shear pressure $\pi$ vanishes rapidly indicating approach to ideal fluid dynamics, the $P_L/P_T$ is far from unity. Faster isotropization for initial $a > 0$ may be attributed to a smaller effective shear viscosity in the modified NS equation.

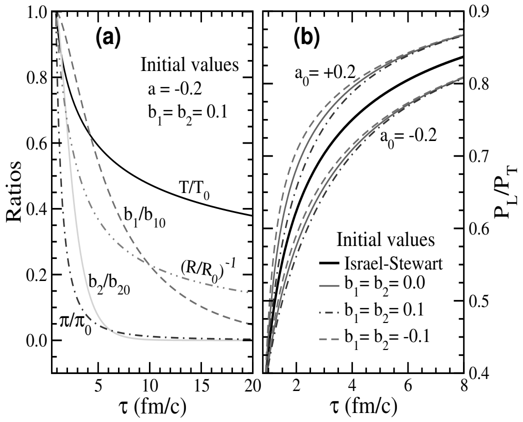


FIG. 1: Time evolution of (a) temperature, shear pressure, inverse Reynolds number and parameters ($b_1$, $b_2$) normalized to their initial values, and (b) anisotropy parameter $P_L/P_T$. Initial values are $\tau_0 = 0.9$ fm/c, $T_0 = 360$ MeV, $\eta/s = 0.16$, $\pi_0 = 4\eta/(3\tau_0)$. Units of $b_2$ are GeV$^{-2}$.

To summarize, we have derived viscous hydrodynamic equations by introducing a nonlocal generalization of the collision term in the Boltzmann equation. The Navier-Stokes and Israel-Stewart equations are modified and new terms are obtained in Eq. (7). The method presented is able to generate all possible terms that are allowed by symmetry. Within 1-dimensional scaling expansion, we find that this modifications has a rather strong influence on the evolution of the viscous medium.

## References

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