

# Random Matrix Theory in Nuclear Structure: Past, present and Future

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Random matrix theory (RMT) introduced by Wigner in 50's to describe statistical properties of slow-neutron resonances in heavy nuclei such as <sup>232</sup>Th, was developed further in the 60's by Dyson, Mehta, Porter and others and in the 70's by French, Pandey, Bohigas and others. Going beyond this, the demonstration that level fluctuations of quantum analogues of classically chaotic few-degrees-of-freedom systems follow random matrix theory (integrable systems follow Poisson as shown by Berry) in 1984 by Bohigas and others on one hand and the recognition from 1995 onwards that two-body random matrix ensembles derived from shell model have wide ranging applications on the other, defined new directions in RMT applications in nuclear physics. Growth points in RMT in nuclear physics are: (i) analysis of nuclear data looking for order-chaos transitions and symmetry (Time-reversal, Parity, Isospin) breaking; (ii) analysis of shell model driven embedded (or two-body) random matrix ensembles giving statistical properties generated by random interactions in the presence of a mean-field; (iii) statistical nuclear spectroscopy generated by embedded ensembles for level densities, occupancies, GT strengths, transition strength sums and so on; (iv) the new paradigm of regular structures generated by random interactions as brought out by studies using various nuclear models; (v) random matrix theory for nuclear reactions with particular reference to open quantum systems; (vi) RMT results from nuclear physics to atomic physics, mesoscopic physics and quantum information science. Topics (i)-(vi) emphasizing recent results are discussed.

## 1. Introduction

Wigner in 1955 introduced random matrix ensembles in physics in his quest to derive information about level and strength fluctuations in compound nucleus resonances. As stated by Wigner: *The assumption is that the Hamiltonian which governs the behavior of a complicated system is a random symmetric matrix with no particular properties except for its symmetric nature.* Further, as French adds: *with one short step beyond this, specifically replacing "complicated" by "non-integrable", this paper would have led to the foundations of quantum chaos. Perhaps it should be so regarded even as it stands.* Dyson gave the tripartite classification of random matrix ensembles giving the classical random matrix ensembles, the Gaussian orthogonal (GOE), unitary (GUE) and symplectic (GSE) ensembles. Porter's book [1] gives an excellent introduc-

tion to random matrix theory and also a collection of papers published till 1965. Similarly, Mehta's book [2], first published in 1967 describes mathematical foundations of classical ensembles.

Significant results of GOE (similarly for GUE and GSE) are: (i) the nearest neighbor spacing ( $S$ ) distribution (NNSD)  $P(S) dS$  (of unfolded spectra) is well represented by the Wigner's surmise  $P(S) dS \sim S e^{-S^2} dS$  (but not by the Poisson law  $e^{-S} dS$ ) showing level repulsion; (ii) the Dyson-Mehta  $\Delta_3$  spectral statistic showing spectral rigidity; (iii) locally renormalized transition strengths ( $x$ ) obey the Porter-Thomas (P-T) law  $P(x) dx \sim x^{-1/2} e^{-x} dx$ . Random Matrix Theory (RMT) has become a common theme in quantum physics with the recognition, by Bohigas and collaborators in 1984, that quantum systems whose classical analogues are chaotic, follow RMT. As summarized by Altshuler in the abstract of the colloquium he gave in memory of J.B. French at the university of Rochester in 2004: "Classical dynamical systems can

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be separated into two classes - integrable and chaotic. For quantum systems this distinction manifests itself, e.g. in spectral statistics. Roughly speaking integrability leads to Poisson distribution for the energies while chaos implies Wigner-Dyson statistics of levels, which are characteristic for the ensemble of random matrices. The onset of chaotic behavior for a rather broad class of systems can be understood as a delocalization in the space of quantum numbers that characterize the original integrable system. . . . Book by Haake [3] is a good reference for all this. With the revival of interest in RMT in physics from 1984, there are now applications of RMT to many diverse fields such as quantum information science, Econophysics, multivariate statistics, information theory, wireless communication, neural networks, biological networks, number theory and so on [4, 5].

In the middle of all these exciting developments in RMT, a new class of random matrix ensembles, called embedded random matrix ensembles have started receiving special attention in quantum physics [6–8]. Isolated finite many-particle quantum systems such as nuclei, atoms, quantum dots, small metallic grains, spin models for quantum computer core, BEC etc. share one common property - *their constituents interact via interactions of low body rank and they are mostly two-body in nature*. Besides this, the particles move in a mean-field giving a one-body term in the Hamiltonian operator. However, representation of the many-particle Hamiltonian by classical ensembles imply many-body interactions. In fact: *the GOE, now almost universally regarded as a model for a corresponding chaotic system is an ensemble of multi-body, not two-body interactions. This difference shows up both in one-point (density of states) and two-point (fluctuations, smoothed transition strengths) functions generated by nuclear shell model*. Two-body interactions imply that many of the many-particle Hamiltonian matrix elements should be zero. Therefore it is more realistic to consider many particle Hamiltonian matrix ensembles generated by random interactions. As here a classical en-

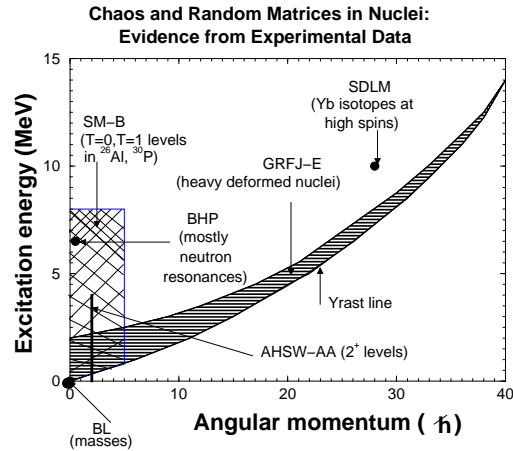


FIG. 1: Schematic diagram giving the regions, in the excitation energy vs angular momentum plane for nuclei, where data was analyzed for evidence for chaos and random matrices.

semble is embedded in many-particle spaces, these are generically called *embedded ensembles* (EE) or random interaction matrix models (RIMM). Most recent reviews on applications of RMT in nuclear physics are given in [8–10]. Now we will give a preview.

In sections 2-7 we will briefly discuss (i) analysis of nuclear data looking for order-chaos transitions and symmetry (Time-reversal, Parity, Isospin) breaking, (ii) shell model driven embedded (or two-body) random matrix ensembles giving statistical properties generated by random interactions in the presence of a mean-field, (iii) statistical nuclear spectroscopy generated by embedded ensembles for level densities, occupancies, GT strengths, strength sums and so on, (iv) the new paradigm of regular structures generated by random interactions as brought out by studies using various nuclear models, (v) random matrix theory for nuclear reactions with particular reference to open quantum systems, and (vi) RMT results from nuclear physics to atomic physics, mesoscopic physics and quantum information science (QIS) respectively. Section 8 gives conclusions.

## 2. Analysis of nuclear data for order-chaos transitions and symmetry breaking

First evidence for the applicability of RMT has come from neutron resonance spacings (testing GOE) and widths (testing P-T). Best analysis for this has been done by Bohigas, Haq and Pandey (BHP) during 1982-85 with 1762 resonance energies corresponding to 36 sequences of 32 different heavy nuclei and 1182 widths corresponding to 21 sequences of the same nuclei. This analysis has been used by French et al to derive, via GOE to GUE transition curve for the number variance, a bound on time reversal symmetry breaking part of the nucleon-nucleon interaction. Garrett et al (GRFJ) analyzed NNSD for high-spin levels near the yrast line in rare-earth nuclei and found, as these levels are regular, Poisson fluctuations. Similarly, Enders (E) et al analyzed NNSD for scissors mode levels in 13 nuclei and also electric pigmy dipole resonances located around 5-7 MeV in four  $N=82$  isotones. This analysis brought out the difference between the scissor mode and pigmy dipole resonance. Using  $2 \times 2$  block matrix GOE with a different variance for the off-diagonal block, Shriner, Mitchell and Barbosa (SM-B) have determined bounds on isospin breaking in  $^{26}\text{Al}$  and  $^{30}\text{P}$ . Using the Rosenzweig-Porter model of drawing levels from several GOE's (fraction of levels from each of them say  $f$ ), Abul-Magd with the Heidelberg group (AWSH) determined best possible values for the chaoticity parameter  $f$  for low-lying  $2^+$  levels and similarly Abul-Magd and Al-Sayed (AA) applied this to prolate vs oblate nuclei. Stephens et al (SDLM) developed a novel technique to measure the chaoticity parameter ( $\Lambda^{1/2}$ ) for order-chaos transition in rotational nuclei. With  $d$  giving the average spacing of the levels that are mixed and  $v$  giving the r.m.s. admixing matrix element,  $\Lambda^{1/2} = v/d$ . NNSD here is given by a simple  $2 \times 2$  matrix for Poisson to GOE transition. Finally, Bohigas and Leboeuf (BL) showed that masses of nuclei exhibit features of chaos. Fig. 1 shows the data analysis carried out so far in  $E-J$  plane. In addition to

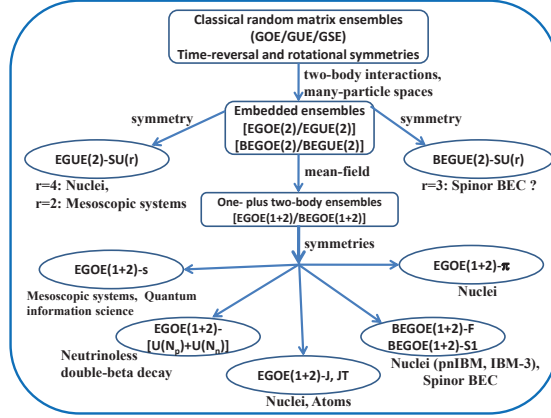


FIG. 2: Embedded random matrix ensembles with symmetries. EGOE/EGUE correspond to fermionic systems and BEGOE/BEGUE correspond to bosonic systems. See [11] for details.

experimental data, there were several studies using nuclear models such as shell model, IBM and other models [8, 9]. Although the analysis using data and models gave insights into order-chaos transitions and symmetry breaking in nuclei, a clear understanding of  $(E, J)$  dependence for a given nucleus for the onset of chaos (or RMT behavior) is not yet available.

## 3. Embedded random matrix ensembles from shell model

Calculations in late 60's with the then new Rochester - Oak Ridge shell mode code by French's group have shown that the smoothed (with respect to energy  $E$ ) level densities (fixed- $J$  or  $JT$  density of eigenvalues)  $I(E)$  take Gaussian form while classical ensembles give Semi-circle form. In the 80's it is found that smoothed transition strengths follow bivariate (in the two energies involved) Gaussian form while classical ensembles give constant value. Similarly, it is seen that chaos measures such as number of principle components (NPC) and information entropy ( $S^{info}$ ) in shell model wavefunctions have quite different behavior compared to GOE results. On the other hand level and strength fluctuations,

after unfolding individual spectra, are seen to follow GOE. Also shell model is seen to generate separation in averages (smoothed forms) and fluctuations and cross correlations (absent in GOE) in spectra with different quantum numbers. Thus both one and two-point functions are different for shell model. These are well verified extensively in both in 2s1d and 2p1f shell examples. All these difference show that the two-body nature of nucleon-nucleon interaction need to be taken into account in RMT. Shell model with ensembles of random two-body interactions is seen to produce the forms for various quantities seen in shell model calculations with realistic interactions. Random matrix ensembles generated by random interactions are called generically embedded random matrix ensembles or simply embedded ensembles (EE).

Starting with a two-body interaction, representing the Hamiltonian matrix in two-particle spaces by GOE and then generating the many particle Hamiltonian for each member of the GOE in two-particle spaces, we have an ensemble of random matrices in many particle spaces. In this random matrix ensemble in many-particle spaces, as a GOE is embedded, this is called EGOE(2); here 'E' stands for embedded and '2' is for two-body. It is important to recognize that for EE there will be correlations between many particle  $H$  matrix elements and they are responsible for generating results different from GOE. The  $H$  operator can have a wide variety of symmetries such as spin ( $s = 1/2$ ), spin-isospin  $SU(4)$ , parity ( $\pi$ ) etc for fermion systems or the fermions can be spinless. These give EGOE(2), EGOE(2-s), EGOE(2)- $SU(4)$ , EGOE(2)- $\pi$  and so on. Similarly, for boson systems it is possible that  $H$  operator carry  $F$ -spin (as in proton-neutron IBM) or spin 1 (as isospin  $T = 1$  in IBM-3 model) degree of freedom or the bosons can be spinless. Then we have BEGOE(2), BEGOE(2)- $F$  and BEGOE(2)- $S1$  ensembles ('B' stands for boson). In addition to two-body interactions, realistic systems also have a mean-field one-body part in the Hamiltonian. The one and two-body parts are denoted by  $h(1)$  and  $V(2)$  respectively. We assume that

$V(2)$  in particle spaces is represented by GOE (it is also possible to consider GUE representation and then we have EGUE and similarly EGSE) with matrix elements variance unity (it is 2 for diagonal matrix elements). Similarly,  $h(1)$  is defined by single particle (sp) energies. We assume that the sp energies have average spacing  $\Delta$ . Then the Hamiltonian is

$$H(1+2) = h(1) + \lambda V(2). \quad (1)$$

Here  $\lambda$  is the strength of the interaction in units of  $\Delta$  and we set, without loss of generality  $\Delta = 1$ . With  $H(1+2)$ , we have EGOE(1+2), EGOE(1+2)-s etc. and these are one plus two-body embedded random matrix ensembles. Fig. 2 shows various EE and the quantum systems where they are applicable. As seen from the figure, EE go beyond nuclear physics and also, they will describe complete statistical behavior rather than just level and strength fluctuations. EEs generate many interesting properties and here below we will discuss one aspect that is important in nuclear structure.

Let us say that the dimension of  $m$  particle space is  $d$  and denote the normalized eigenvalue density by  $\rho(E)$  and total density by  $I(E)$ . With  $\langle \rangle$  and  $\langle\langle \rangle\rangle$  denoting average and trace over a given space,

$$\begin{aligned} \rho(E) &= \langle \delta(H - E) \rangle, \\ I(E) &= \langle\langle \delta(H - E) \rangle\rangle = d \rho(E). \end{aligned} \quad (2)$$

Expanding eigenfunctions  $|E\rangle$  in terms of some basis states (they form a complete set)  $|k\rangle$ , we have  $|E\rangle = \sum_{k=1}^d C_k^E |k\rangle$ . Then, strength functions or local density of states  $F_k(E)$ , giving the spreading of the basis states  $|k\rangle$  over the eigenstates, are defined by,

$$F_k(E) = \sum_{E'} \left| C_k^{E'} \right|^2 \delta(E - E') = |C_k^E|^2 I(E) \quad (3)$$

where  $|C_k^E|^2$  denotes the average of  $|C_k^E|^2$  over the eigenstates with the same energy  $E$ . Similarly, NPC (denoted as  $\xi_2$ ) - number of principal components and  $S^{info}$  - information entropy or  $\ell_H$  - localization length are defined

by,

$$\begin{aligned} \xi_2(E) &= \left[ \frac{1}{d\rho(E)} \sum_{E'} \sum_{k=1}^d |C_k^{E'}|^4 \delta(E - E') \right]^{-1}, \\ \ell_H(E) &= \exp \left[ (S^{info})_E \right] / (0.48d), \\ (S^{info}(E) &= -\frac{1}{d\rho(E)} \sum_{E'} \sum_{k=1}^d |C_k^{E'}|^2 \\ &\times \ln |C_k^{E'}|^2 \delta(E - E'). \end{aligned} \quad (4)$$

Starting with the EGOE(1+2) Hamiltonian defined by Eq. (1) and increasing  $\lambda$  value from zero, the following are observed:

1. Eigenvalue density will be essentially of Gaussian form for all  $\lambda$  values.
2. As  $\lambda$  increases, there is a transition from Poisson to GOE fluctuations with the onset of GOE fluctuations at  $\lambda = \lambda_c$ .
3. For  $\lambda \sim 0$  strength functions will be delta functions and then quickly turn into Breit-Wigner (BW) form at  $\lambda = \lambda_0$  with  $\lambda_0 \ll \lambda_c$ . As  $\lambda$  increases beyond  $\lambda_c$  there will be a transition from BW form to Gaussian with the onset of this transition at  $\lambda = \lambda_F > \lambda_c$ .
4. As we increase  $\lambda$  further, there will be a region around  $\lambda \sim \lambda_t > \lambda_F$  where different definitions of entropy, temperature etc. will coincide defining 'thermodynamic region'.

Existence of the three chaos or transition markers  $\lambda_c$ ,  $\lambda_F$  and  $\lambda_t$  has been established numerically for both fermion and boson systems by analyzing spinless and spin EE. Same structure is also seen in shell model calculations with random two-body interactions having  $J$  or  $JT$  symmetry [EGOE(1+2)- $J$  or EGOE(1+2)- $JT$ ] and more importantly, also with realistic interactions in presence of a mean-field by changing all the two-body matrix elements by a factor. These results define statistical spectroscopy or spectral distribution methods for nuclei (also for atoms) and this discussed in Section 4. In addition, they will allow us to investigate thermalization

in isolated finite quantum systems, a topic of great current interest in quantum physics (see Section 7).

#### 4. Statistical spectroscopy for nuclear structure

Nuclear shell model examples have shown that realistic sp energies and effective interactions that are being used are such that nuclear systems are in  $\lambda \sim \lambda_t$  region. Therefore, for complex nuclei smoothed level densities, strength functions [as the total sum of strength function over  $k$  will give  $I(E)$ , these or their partial sums are called partial densities in nuclear structure applications], transition strength densities etc. will take Gaussian form and local fluctuations follow GOE. Note that, given a transition operator  $\mathcal{O}$  the transition strength density  $I_{\mathcal{O}}(E_i, E_f) = I(E_f) |\langle E_f | \mathcal{O} | E_i \rangle|^2 I(E_i)$ . These Gaussian forms (valid to a large extent even for  $\lambda_t > \lambda > \lambda_F$ ) given by EGOE(1+2)s can be used to calculate spectroscopic quantities such level densities, orbit occupancies, strength sums (GT, E2 and M1), strength distributions (for example GT), beta decay and double beta decay half lives and so on. As fluctuations follow GOE, they will be small (operating over a few mean spacings) and hence can be neglected for most purposes. Six important aspects that should be recognized are as follows. (i) it is assumed that EGOE(1+2) and EGOE(1+2)-s results extend to EGOE(1+2) with shell model  $J$  or  $JT$  symmetries and this is well verified by large shell model (questions remain for high  $J$  states); (ii) parameters defining the forms given by EGOE(1+2)s can be calculated without  $H$  matrix construction as they are 'moments' and moments being traces of products of operators, it is possible to write propagation equations for them; (iii) as shell model spaces are large and convergence to Gaussian form is poor in the tails, it is necessary to partition the spaces to subspaces  $\Gamma$  and apply EGOE results in the subspaces. For example the partial densities  $I^\Gamma(E) = \langle \langle \delta(H - E) \rangle \rangle^\Gamma$  will be Gaussian as they are sum of strength functions. In practice  $\Gamma$  correspond to shell model proton-neutron configurations defined

by distributing valence protons and neutrons in their respective shell model orbits. Then for example,  $I(E) = \sum_{\Gamma} I_{ED}^{\Gamma}(E)$  where 'ED' is Edgeworth corrected Gaussian that includes third and fourth moment corrections; (iv) usually number of  $\Gamma$ 's will be large and to a good approximation, the sums over  $\Gamma$  can be replaced by 'convolutions'; (v) it is possible to directly calculate all the densities over fixed- $J$  or  $JT$  spaces and alternatively  $J$  can be projected using energy dependent spin-cutoff factors; (vi) ground state energy ( $E_{GS}$ ) can be determined by using the so called Ratcliff prescription or the exponential convergence method of Zelevinsky. There are various ways (i)-(vi) can be applied giving statistical spectroscopy approach or spectral distribution method for nuclear structure. This subject was described in detail in a recent book [12]. Tables I and II give a list of applications and references for them.

TABLE I: Level densities with interactions

Method, Authors and References
<b>Convolution forms, partitioning, <math>J</math> via spin-cutoff</b> D. Majumdar, <i>Nucl. Phys.</i> <b>A604</b> (1996) 129 J.B. French, S. Rab, J.F. Smith, R.U. Haq and V.K.B. Kota, <i>Can. J. Phys.</i> <b>84</b> (2006) 677
<b>Fixed-<math>J</math> densities with partitioning</b> M. Horoi, J. Kaiser and V. Zelevinsky, <i>Phys. Rev. C</i> <b>67</b> (2003) 054309 M. Horoi, M. Ghita and V. Zelevinsky, <i>Phys. Rev. C</i> <b>69</b> (2004) 041307(R)
<b>Fixed-<math>J</math> densities with partitioning, center of mass correction, high-performance computing</b> R.A. Senkov and M. Horoi, <i>Phys. Rev. C</i> <b>82</b> (2010) 024304 (2010). R.A. Senkov, M. Horoi and V.G. Zelevinsky, <i>Phys. Lett.</i> <b>B702</b> (2011) 413 (2011)

## 5. Regular structures from random interactions

Johnson et al discovered in 1998, using numerical experiments, that the nuclear shell model with random two-body interactions generates, with high probability,  $0^+$  ground states in even-even nuclei and also generates odd-even staggering in binding energies and the seniority pairing gap. Similarly, Bijker and Frank in 2000 found that the interacting boson model (*sd*IBM) with random interactions generates vibrational and rotational

TABLE II: Applications of Transition Strengths Theory in Nuclear Structure.

No.	Topic	Authors and References
1	Bound on time reversal non-invariant part of nucleon-nucleon interaction	J.B. French, V.K.B. Kota, A. Pandey and S. Tomsovic <i>Phys. Rev. Lett.</i> <b>58</b> (1987) 2400 <i>Ann. Phys. (N.Y.)</i> <b>181</b> (1988) 235
2	Parity breaking matrix elements in compound resonance region	S. Tomsovic, M.B. Johnson, A.C. Hayes and J.D. Bowman <i>Phys. Rev. C</i> <b>62</b> (2003) 054607
3	Single particle transfer	V. Potbhare and N. Tressler <i>Nucl. Phys.</i> <b>A530</b> (1991) 171
4	Beta decay half lives and rates for presupernovae stars and r-process	K. Kar, S. Sarkar and A.Ray <i>APJ</i> <b>434</b> (1994) 662 V.K.B. Kota and D. Majumdar <i>Z. Phys.</i> <b>A351</b> (1995) 377 K. Kar, S. Chakravarti and V.R. Manfredi <i>Pramana-J. Phys.</i> <b>67</b> (2006) 363
5	Giant dipole widths	D. Majumdar, K. Kar and A. Ansari <i>J. Phys.</i> <b>G23</b> (1997) L41; <b>G24</b> (1998) 2103
6	Double beta decay	Manan Vyas and V.K.B. Kota arXiv:1106.0395 [nucl-th] (2011)

structures with high probability. Starting with these, there are now many studies on regular structures in many-body systems generated by random interactions. For example some of the questions addressed are: (i) why ground states of all even-even nuclei have positive parity; (ii) why there is preponderance of prolate shape compared to oblate shape; (iii) why there is natural isospin ordering - denoting the lowest eigenvalue state (*les*) for a given many nucleon isospin  $T$  by  $E_{les}(T)$ , the natural isospin ordering corresponds to  $E_{les}(T_{min}) \leq E_{les}(T_{min} + 1) \leq \dots$  (for even-even  $N=Z$  nuclei,  $T_{min} = 0$ ); (iv) for proton-neutron boson systems with  $F$ -spin, why there is predominance of maximum  $F = F_{max}$  (i.e. IBM-1) ground states; (v) why quadrupole collectivity dominates. Shell model and IBM with random interactions are found to generate all these regular structure. This is a new paradigm and early reviews are in [13, 14]. Some of these numerical results are understood using four different approaches.

Zhao et al proposed an empirical approach to predict  $P(J)$  the probability to have ground spin  $J$  with random two-body interactions. Energies of  $J$  states [say there are  $\beta$  number of eigenvalues with same  $J$  and denote the

two-body matrix elements by  $G(J_2, \alpha)$  where  $J_2$  is two particle  $J$  and  $\alpha$  is number of two particle states with same  $J_2$ ] are given by  $E(J, \beta) = \sum_{J_2, \alpha} \mathcal{C}(J, \beta : J_2, \alpha) V(J_2, \alpha)$ . In some simple situations it is possible to examine the geometric factors  $\mathcal{C}$ 's and from them infer  $P(J)$ . For single  $j$  shell for fermions, single  $\ell$  shell for bosons, and also in some situations for multi shell examples the empirical approach is shown to apply [14].

Second approach, first used by Bijker and Frank and later generalized by Kota [15] for boson systems is to employ mean-field approach. Given  $N$  bosons in  $\mathbf{n}$  single particle states, the SGA for a IBM is  $U(\mathbf{n})$  and all states belong to the symmetric irrep  $\{N\}$  of  $U(\mathbf{n})$ . Considering random Hamiltonians (maximum two-body) that are  $SO(\mathbf{n}_1)$  and  $SO(\mathbf{n}_2)$  scalars in  $U(\mathbf{n}) \supset G \supset SO(\mathbf{n}_1) \oplus SO(\mathbf{n}_2) \supset K$  with  $\mathbf{n}_1 + \mathbf{n}_2 = \mathbf{n}$ , which appears in many extensions of IBM-1, and using the mean-field approach addressed is the question of with what probability a given  $(\omega_1 \omega_2)$  irreducible representation (irrep) will be the ground state in even-even nuclei;  $[\omega_1]$  and  $[\omega_2]$  are the symmetric irreps of  $SO(\mathbf{n}_1)$  and  $SO(\mathbf{n}_2)$  respectively. We will restrict to  $\mathbf{n}_1 \geq 3$  and  $\mathbf{n}_2 \geq 3$ . For this system there are two  $G$ 's possible: (i)  $U(\mathbf{n}_1) \oplus U(\mathbf{n}_2)$  and (ii)  $SO(\mathbf{n})$ . In terms of the linear and quadratic Casimir invariants, general  $H$  for this system will have 8 parameters,. Generating random  $H$ 's by choosing these parameters to be zero centered independent Gaussian random variables with unit variance, for a 500 member ensemble, calculations have been carried out by constructing and diagonalizing the  $H$  matrix for boson numbers  $N = 10 - 25$ . Results are shown in Table III. To explain these results, considered is the energy functional  $E(\alpha)$ , for a simple  $H$  interpolating the two group chains, evaluated over a  $N$  boson coherent state (CS). The CS is defined in terms of the intrinsic bosons  $y_0^\dagger = \frac{1}{\sqrt{p}} \sum_{i=1}^p b_{\ell_i, 0}^\dagger$ ,  $\sum_{i=1}^p (2\ell_i + 1) = \mathbf{n}_1$  and  $z_0^\dagger = \frac{1}{\sqrt{q}} \sum_{j=1}^q b_{\ell'_j, 0}^\dagger$ ,  $\sum_{j=1}^q (2\ell'_j + 1) = \mathbf{n}_2$ . Then,

$$H = \frac{1}{N} \cos \chi \hat{n}_2 + \frac{1}{N(N-1)} \sin \chi S_+ S_-$$

$$S_+ = \sum_{i=1}^p b_{\ell_i}^\dagger \cdot b_{\ell_i}^\dagger - \sum_{j=1}^q b_{\ell'_j}^\dagger \cdot b_{\ell'_j}^\dagger; \quad S_- = (S_+)^\dagger$$

$$|N \alpha\rangle = \frac{1}{\sqrt{N!}} \left( \cos \alpha y_0^\dagger + \sin \alpha z_0^\dagger \right)^N |0\rangle$$

$$E(\alpha) = \langle H \rangle^{N, \alpha} = \cos \chi \sin^2 \alpha + \frac{1}{4} \sin \chi \cos^2 2\alpha. \quad (5)$$

Minimizing  $E(\alpha)$  gives three solutions:  $\alpha = 0$ ,  $\alpha = \pi/2$  and  $\cos 2\alpha = \cot \chi$ . For  $\alpha = 0$  we have  $y$ -boson condensate with energy  $E(\alpha = 0) \propto -\sin \chi \omega_1 (\omega_1 + \mathbf{n}_1 - 2)$ . Then the ground state irreps are  $(\omega_1 \omega_2) = (00)$  with 25% and  $(\omega_1 \omega_2) = (N0)$  with 12.5% probability. Similarly,  $\alpha = \pi/2$  gives  $z$ -boson condensate with energy  $E(\alpha = \pi/2) \propto -\sin \chi \omega_2 (\omega_2 + \mathbf{n}_2 - 2)$  and then the ground state irreps are  $(\omega_1 \omega_2) = (00)$  with 25% and  $(\omega_1 \omega_2) = (0N)$  with 12.5% probability. In the situation  $\cos 2\alpha = \cot \chi$ , cranking has to be done with respect to both  $SO(\mathbf{n}_1)$  and  $SO(\mathbf{n}_2)$ . Evaluating moment of inertia, by an extension of the ordinary  $SO(3)$  cranking, gives  $(N0)$  and  $(0N)$  irreps will be ground states each with 12.5% probability. Combining all the results give,  $(\omega_1 \omega_2) = (00)$ ,  $(N0)$  and  $(0N)$  irreps to be ground states with 50%, 25% and 25% probability in complete agreement with numerical findings in Table III.

TABLE III: Probabilities (in percentage) for  $(\omega_1 \omega_2)$  to be ground state irrep

Model	$\mathbf{n}_1$	$\mathbf{n}_2$	$N$	$(\omega_1 \omega_2) = (00)$	$(\omega_1 \omega_2) = (N0)$	$(\omega_1 \omega_2) = (0N)$		
$U(7)$	4	3	10	55.4	21.4	20.5		
			20	54	21.2	20.6		
$spd$ IBM	6	3	10	55	22.4	19.8		
			20	53.5	21.9	20.3		
$sdg$ IBM	10	5	10	55.3	22.9	19.2		
			20	53.6	22.8	20.4		
$spdf$ IBM	10	6	10	49.3	24.6	21.9		
			20	49.3	23.8	22		
$sdgpf$ IBM	15	10	10	53.8	22.9	20.3		
			20	54.4	22.4	20.5		
IBM-3	15	3	10	49	27.1	19.8		
			20	49	25.6	20.7		
IBM-4	3	3	10	49.2	22.8	22.8		
			20	48.8	23.2	23.2		
			30	6	10	50.1	28.6	18.6
			18	18	10	50	23.5	23.5
	15	15	10	49.6	23.6	23.6		

In a third approach energy centroids and spectral variances over various spaces for fermion and boson systems are analyzed using random interactions. This is feasible as propagation equations (exact or approximate) can be written for these and therefore they can

be calculated without  $H$  matrix construction. For example for  $m$  fermions in a single  $j$  shell with three body interactions, the energy centroids for angular momentum  $I$  states are of the form  $\overline{E_{(m,I)}} \simeq E_0 + m f(j, G_{\alpha,J}) I(I+1)$ ;  $G$  are TBME. Distribution of energy centroids and spectral variances give information about regular structures [7, 9, 13, 14] and Zelevinsky calls this 'geometric chaos'.

In the fourth approach EGOE(1+2)-s and BEGOE(1+2)- $F$  [also EGUE(2)- $SU(4)$ , BEGUE(2)- $SU(3)$ ] are used by adding exchange and pairing terms to the Hamiltonian. In the strong coupling region, ensemble averaged spectral variances show that ground states will have spin  $S = 0$  for fermions ( $F = F_{max}$  for bosons) and similarly there will be odd-even staggering in ground state energies. See [8, 11] for details.

## 6. Random matrix theory for open quantum systems: basic results

RMT is usually applied to isolated (closed) finite quantum systems where the coupling to the environment can be neglected. However, compound nuclear resonances and also many systems of current interest such as quantum dots, micro lasers cavities, microwave billiards etc., coupling of the quantum system to the environment must be explicitly taken into account. Properties of the open and marginally stable quantum many-body systems can be studied in a general fashion using the effective Hamiltonian with a complex part of the type [16],  $H_{eff} = H_0 - \frac{i}{2} VV^\dagger$ ;  $H_0$  gives the discrete spectrum and  $VV^\dagger$  represents the coupling to the continuum. With  $N$  discrete states coupled to  $M$  open channels ( $N \gg M$ ),  $H_0$  is a  $N \times N$  matrix and  $V$  is a  $N \times M$  matrix. We will restrict the discussion to time-reversal and rotationally invariant systems. Therefore,  $H_0$  is real symmetric matrix and similarly, the matrix elements of  $V$  are real. Thus, in the random matrix approach,  $H_{eff}$  will be a random matrix ensemble with  $H_0$  represented by a GOE and  $V$  matrix elements are chosen to be independent Gaussian variables with zero

center and variance say  $1/\eta$ ,

$$\{H_{eff}\} = \{H_0\} - \frac{i}{2} \{VV^\dagger\} \quad (6)$$

where  $\{-\}$  represents ensemble. Due to the  $V$  part, the eigenvalues of  $H_{eff}$  will be complex and we can write them as  $E - \frac{i}{2}\Gamma$ . For example for resonance states,  $E$  represents the resonance positions and  $\Gamma$  their width. Using the ensemble defined by Eq. (6) one can study for example the statistics of neutron resonance spacings in the region where the resonance widths are not very small compared to the average resonance spacing and similarly the modifications to the resonance width distribution (i.e. modification to P-T law).

Let us consider the simplest situation of  $N = 2$  and  $M = 1$ , i.e. two bound states coupled to a single open channel. Then,  $H_{eff}$  (hereafter we call it  $H$ ) matrix structure is,

$$H = \begin{pmatrix} a & b \\ b & c \end{pmatrix} - i \begin{pmatrix} x_1^2 & x_1 x_2 \\ x_1 x_2 & x_2^2 \end{pmatrix} \quad (7)$$

where  $a, b, c, x_1$  and  $x_2$  are independent  $G(0, 2v^2)$ ,  $G(0, v^2)$ ,  $G(0, 2v^2)$ ,  $G(0, \sigma^2)$  and  $G(0, \sigma^2)$  variables. In the  $H_0$  diagonal basis, with  $E_1^0$  and  $E_2^0$  being its eigenvalues, the structure of  $V$  will remain unaltered, i.e.  $(x_1, x_2) \rightarrow (X_1, X_2)$  with  $X_1$  and  $X_2$  being independent  $G(0, \sigma^2)$  variables. Then,

$$H = \begin{pmatrix} E_1^0 & 0 \\ 0 & E_2^0 \end{pmatrix} - i \begin{pmatrix} X_1^2 & X_1 X_2 \\ X_1 X_2 & X_2^2 \end{pmatrix} \quad (8)$$

Let us define  $c_1 = X_1^2$  and  $c_2 = X_2^2$ . Then it is easy to see that the joint probability distribution  $P(E_1^0, E_2^0, c_1, c_2)$  is,

$$P(E_1^0, E_2^0, c_1, c_2) \propto \frac{|E_1^0 - E_2^0|}{\sqrt{c_1 c_2}} \exp - \left\{ \frac{(E_1^0)^2 + (E_2^0)^2}{4v^2} + \frac{c_1 + c_2}{2\sigma^2} \right\}. \quad (9)$$

Denoting the two eigenvalues of  $H$  as  $\mathcal{E}_1 = E_1^R + iE_1^I$  and  $\mathcal{E}_2 = E_2^R + iE_2^I$ , to derive the joint distribution  $P(E_1^R, E_2^R, E_1^I, E_2^I)$  we need the Jacobi determinant

$$\left| \frac{\partial(E_1^0, E_2^0, c_1, c_2)}{\partial(E_1^R, E_2^R, E_1^I, E_2^I)} \right|.$$



Diagonalizing the matrix given by Eq. (8), formulas for  $\mathcal{E}_1$  and  $\mathcal{E}_2$  in terms of  $(E_1^0, E_2^0, c_1, c_2)$  are easily obtained. Now equating the real and imaginary parts of the sum of the eigenvalues and similarly the sum of the squares of the eigenvalues will give,

$$X = E_1^0 + E_2^0 = E_1^R + E_2^R, \\ Y^2 = (E_1^0 - E_2^0)^2 = (E_1^R - E_2^R)^2 + 4E_1^I E_2^I,$$

$$c_i = -\frac{E_1^I + E_2^I}{2} + (-1)^i \frac{(E_1^R - E_2^R)(E_1^I - E_2^I)}{2Y}.$$

In the variables  $(X, Y, c_1, c_2)$ , the Jacobi determinant is  $-(2/Y^2)[(E_1^R - E_2^R)^2 + (E_1^I - E_2^I)^2]$ . Combing this with Eqs. (9) will give  $P(E_1, E_2, \Gamma_1, \Gamma_2)$ ; defining the eigenvalues of  $H$  to be  $\mathcal{E}_i = E_i - \frac{i}{2}\Gamma_i$ ,  $i = 1, 2$  we have  $E_i^R = E_i$  and  $E_i^I = -\Gamma_i/2$ . Putting  $A = 1/4v^2$  and  $\sigma^2 = 1/2\eta$ , we have,

$$P(E_1, E_2, \Gamma_1, \Gamma_2) \propto \frac{1}{[(E_1 - E_2)^2 + \frac{1}{4}(\Gamma_1 - \Gamma_2)^2]} \frac{1}{[(E_1 - E_2)^2 + \frac{1}{4}(\Gamma_1 + \Gamma_2)^2]^{1/2} (\Gamma_1 \Gamma_2)^{1/2}} \\ \times \exp[-A(E_1^2 + E_2^2 + \frac{\Gamma_1 \Gamma_2}{2}) - \frac{\eta}{2}(\Gamma_1 + \Gamma_2)]. \quad (10)$$

Now, changing the variables  $E_1$  and  $E_2$  to  $Z = E_1 + E_2$  and  $s = E_1 - E_2$  and integrating over  $Z$  gives  $P(s, \Gamma_1, \Gamma_2)$ . Further integration over  $\Gamma_1$  and  $\Gamma_2$  will give the distribution of the spacings ( $s$ ) between the real parts of the eigenvalues,

$$P(\hat{S} : \Lambda) d\hat{S} = \mathcal{N} d\hat{S} \exp -\frac{\hat{S}^2}{2} \\ \times \int_0^\infty \left\{ \frac{dx}{\sqrt{\hat{S}^2 + x^2/4}} \left[ \exp\left(-\frac{x^2}{32} - \frac{\Lambda}{2}x\right) \right] \right. \\ \left. \times \left[ (8\hat{S}^2 + x^2) I_0\left(\frac{x^2}{32}\right) + x^2 I_1\left(\frac{x^2}{32}\right) \right] \right\}. \quad (11)$$

In Eq. (11),  $\hat{S} = \sqrt{A}s$ ,  $\Lambda = \eta/\sqrt{A}$  and  $I_n$  are modified Bessel functions of first kind. The constant  $\mathcal{N}$  follows from the normalization condition  $\int_0^\infty P(\hat{S} : \Lambda) d\hat{S} = 1$ . Note that  $\sqrt{A} \propto 1/\Delta$  where  $\Delta$  is the mean level spacing of the closed system (defined by  $H_0$ ). Similarly the transition parameter  $1/\Lambda \propto \bar{\Gamma}/\Delta$  where  $\bar{\Gamma}$  is the average width. In [17] it was shown that Eq. (11) applies to the general situation

with any  $N$  and  $M$  by treating  $\Lambda$  as an effective parameter. Further, Eq. (11) shows that level repulsion is suppressed for open systems as there is finite probability for zero spacings. Thus the real parts of the eigenvalues will be attracted due to the coupling to the environment. The non zero probability for  $s \sim 0$  is clearly seen in open chaotic 2D microwave cavity experiments [17].

Turning to P-T, let us consider the situation with any  $N$  and  $M = 1$  and denote the real part of the  $N$  eigenvalues  $E - \frac{i}{2}\Gamma$  by  $\vec{E} = (E_1, E_2, \dots, E_N)$  and the imaginary parts by  $\vec{\Gamma} = (\Gamma_1, \Gamma_2, \dots, \Gamma_N)$ . The joint distributions  $P(\vec{E}, \vec{\Gamma})$  in the real and imaginary parts of the eigenvalues for any  $N$  and  $M = 1$  can be written by inspection using the  $2 \times 2$  matrix result given by Eq. (10). Using this along with a mean-field approximation, the modified P-T law for the width distribution  $P(\Gamma) d\Gamma$  is [18],

$$P(\Gamma) d\Gamma \propto \frac{1}{\sqrt{(N\gamma - \Gamma)(\Gamma)}} \left( \frac{\sinh \kappa}{\kappa} \right)^{1/2} \\ \times \exp[-(N/2q^2)(N\gamma - \Gamma)\Gamma] d\Gamma. \quad (12)$$

Here  $\kappa = \pi\Gamma/2\bar{D}$  and  $q/N$  determines the mean decay amplitude. Also,  $\bar{D}$  is average spacing between the real parts of the eigenvalues and  $\gamma = \bar{\Gamma}$  is the average width. The first factor in Eq. (12) is essentially P-T and the second factor determines the deviations. Recently, new questions on the applicability of P-T law for slow neutron resonance widths have been raised [19] and the approach discussed here, taking into account the coupling to the continuum explains the source of the deviation from P-T [18].

## 7. EE from nuclei to other quantum systems

Only in the last ten years it is recognized that EE, born in nuclear physics, apply to atomic spectroscopy [20] and this is based on the recognition that the statistical properties seen in large atomic shell model results for complex lanthanide atom Ce I by Flambaum [21] indeed imply operation of statisti-

cal spectroscopy derived from EE. More significantly, EE are found to be useful in the study of certain properties of mesoscopic systems such as quantum dots and small metallic grains [11, 22], for example conductance fluctuations in Coulomb blockade regime in quantum dots, odd-even staggering in binding energies of small metallic grains and delay in Stoner instability in itinerant systems. Finally, very recently it is argued that EE will be useful in quantum information science in general and in understanding thermalization in isolated finite quantum systems in particular [23] with tests using experiments with ultracold gases trapped in optical lattices.

## 8. Conclusions

A brief overview of the progress in RMT in nuclear physics, with focus on EE and new results is given. New nuclear experiments aiming at tests of various predictions and consequences of EE in particular and RMT in general are clearly called for. They will enrich the subject in future.

## Acknowledgments

Thanks are due to my collaborators Manan Vyas, N.D. Chavda, V. Potbhare, J.M.G. Gomez, J. Retamosa, R.Sahu, K. Kar, R.U. Haq and Jinya John.

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