Derivation of a novel formula for $\alpha-$decay half-life

Basudeb Sahu and Swagatika Bhoi*
Department of Physics, College of Engineering and Technology, Bhubaneswar-751003, INDIA and
* Department of Physics, Sambalpur University, Burla-768019, INDIA

Based on the basic principle of quantal decay of particle, we derive a formula of logarithm of decay half-life of an alpha particle emiting from a radioactive nucleus. The process of decay is understood as the transition of the particle from an isolated quasi-bound state to a scattering state. In this picture, the decay width is a resonance width in the system consisting of an α-cluster and the residual nucleus. The general formula of the width can be expressed by [1]

$$\Gamma = 2\pi \, |< \psi \mid H - H_0 \mid \phi >|^2 \qquad (1)$$

Where ψ is a bound initial state for the decaying nucleus, and ϕ is a final scattering state for the α+daughter system. The Hamiltonians H_0 qnd H are associated with ϕ and ψ, respectively.

The final state wave function ϕ is expressed as

$\phi_\ell(r) = \sqrt{\frac{2\mu}{\pi\hbar^2 k}}\ \frac{F_\ell}{r}$, where k=$\sqrt{2\mu E}/\hbar$, μ is the reduced mass of the α+daughter system, and F_ℓ is the regular Coulomb wave function.

The initial state wave function describes the quasi-bound state of α cluster in the decaying nucleus. Its radial wave function, $\psi_{n\ell}(r) = \frac{u_{n\ell}}{r}$, is achieved through exact solution of the Schrödinger equation with an effective potential which is a combination of nuclear potential and the electrostatic potential in the ℓ=0 case [2].

The α cluster in the decaying nucleus is governed by a nuclear potential $V_N(r)$ and a Coulomb potential $V_C(r)$ while the α particle outside the nucleus by the point-charge Coulomb potential $V_C^P(r)=Z_1Z_2e^2/r$. H-H_0 is considered as the difference between the potentials in the two situations, that is,

$$H-H_0 = (V_N(r)+V_C(r))-V_C^P(r) = V_{eff}(r)-V_C^P(r). \qquad (2)$$

Using the exact wave function $u_{n\ell}$(r) for ℓ=0 in the interior region [2], the expression (1) of the decay width reduces to

$$\Gamma = \frac{4\mu}{\hbar^2 k}\frac{\mid \int_0^R F_\ell(V_{eff}(r) - V_c^p(r))u_{n\ell}(r)dr \mid^2}{\int_0^R \mid u_{n\ell}(r) \mid^2 dr}. \qquad (3)$$

Then the α-decay half-life is related to the decay width by the well known relationship T=$\hbar$ $ln2/\Gamma$. Using Γ given by (3) in it, the half-life T is expressed as

$$T = \frac{0.693\ \hbar^3 k}{4\mu}\frac{1}{J}\ , \qquad (4)$$

$$J = \frac{\mid \int_0^R F_\ell(V_{eff}(r) - V_c^p(r))u_{n\ell}(r)dr \mid^2}{\int_0^R \mid u_{n\ell}(r) \mid^2 dr}. \qquad (5)$$

For a typical α+nucleus system with its characteristic energy Q-value, the values of Sommerfeld parameter η and parameter ρ=kR are such that the product $\eta\rho \leq 50$ and $\rho \approx 10$. In this situation, one can express the regular Coulomb wave function F_ℓ(r) by using power series expansion and write as

$$F_\ell^{ps}(r) = C_\ell \rho^{\ell+1} B, \qquad (6)$$

$$(n+1)(n+2\ell+2)B_{n+1} = 2\eta\rho B_n - \rho^2 B_{n-1}, \qquad (7)$$

B_0=1, B_1=$\frac{\eta\rho}{(\ell+1)}$, B=$\sum_j B_j$,

$$C_\ell^2 = \frac{P_\ell(\eta)}{2\eta}\ \frac{C_0^2(\eta)}{(2\ell+1)}, \qquad (8)$$

*Electronic address: bd_sahu@yahoo.com

$$P_\ell(\eta)=\frac{2\eta(1+\eta^2)(4+\eta^2).....(\ell^2+\eta^2)2^{2\ell}}{(2\ell+1)[(2l)!]^2}. \qquad (9)$$

In particular, for ℓ=0, $P_0(\eta)=2\eta$ and

$$C_0^2(\eta)=2\pi\eta\left\{exp(2\pi\eta)-1\right\}^{-1}, \qquad (10)$$

$$F_0^{ps}(r)=C_0\rho\tilde{B}, \qquad (11)$$

where $\tilde{B}$ is equal to B derived through expression (7) taking ℓ=0.

On comparision, we find the exact Coulomb wave function for $\ell=0$, $F_0(r)=x_m\ F_0^{ps}(r)$, with $x_m\approx 70$.

The integral J given by expression (5) can be equated to the value of $F_0(r)$ at a point r=R$\leq\ R_B$ with some multiplying factor to account for the contributions to the integral from other regions within $0<r<R$ such that $J=\mid c_f F_0(R)\mid^2=\mid c_f\ x_m F_0^{ps}(R)\mid^2$, for ℓ=0. The value of $c_f(=J/[x_m F_0^{ps}(R)]^2)$ for a typical α+nucleus system is found to be of the order of 0.22.

Using this simplified result for J and the expression (11) for $F_0^{ps}(R)$ with $C_0^2(\eta)\approx\frac{2\pi\eta}{exp(2\pi\eta)}$ due to large value of $2\pi\eta$, the expression (4) for T reduces to

$$T=0.693\hbar\ \frac{exp(2\pi\eta)}{f_m^2}, \qquad (12)$$

where

$$f_m=\sqrt{4\mu/\hbar^2 k}\sqrt{2\pi\eta}\ c_f\ x_m\ \rho\tilde{B}. \qquad (13)$$

Taking logarithm of both sides and writing η, ρ and f_m explicitly in terms of Z_1, Z_2, A_1, A_2, and Q-value, we get

$$logT=a\ \chi+c+d, \qquad (14)$$

where $a=1.4398\pi Z_1 Z_2\sqrt{2(931.5)}/197.329$, $\chi=\sqrt{\frac{A_1A_2}{(A_1+A_2)\ Q}}$, $c=-2logD$, $d=-2logS$, $D=\frac{2\times931.5\times\sqrt{1.4398\times2\pi}}{(197.329)^2\sqrt{0.693\times197.329\times0.333\times10^{-23}}}$, and $S=c_f x_m R\tilde{B}\left(\frac{A_1A_2\sqrt{Z_1Z_2}}{A_1+A2}\right)$. In our calculation for the potential we have taken R=$r_0(A_1^{0.3333}+A_2^{0.3333})+2.72$ with r_0=0.97 fm that specifies the radial position of the Coulomb barrier, c_f=0.22 and x_m=70.

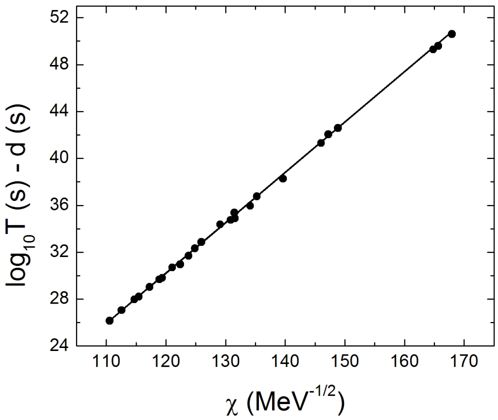

FIG. 1: Plot of decay law (14) for α-decays in $\ell=0$ state from decaying nuclei with Z=60−74. The straight line is given as $a\chi+c$ for ($\log_{10}$T-d) of (14).

As the parameter d depends on energy Q through the quantity $\tilde{B}$, the variation of logT as function of χ does not represent a straight line but the quantity (logT-d) as a function χ does show a linear variation.

We consider a set of naturally existing nuclei for which experimental data of Q and α decay half-lives are available. In Fig. 1, the results from experiments $log_{10}T^{(expt)}-d$[3] and our calculated values $log_{10}T^{(calt)}-d$ are plotted as a function of the quantity χ defined in (14). It is clearly seen that the experimental data (solid dots) which align themselves in a linear path are explained by our calculated results in the form of a perfect straight line remarkably well.

References

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[2] B Sahu, *Phys. Rev.* **C 78** 044608 (2008), *Phys. Rev.* **C 84** 037607 (2011), *Phys. Rev.* **C 85** 057601 (2012).

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