Analytical solutions of the Schroedinger equation with the Woods-Saxon potential for $l = 0$ states

Antony Prakash Monteiro[1]* and Manjunath Bhat[1]
[1]*P.G Department of Physics, St Philomena College, Darbe, Puttur-574202,India*

Introduction

An analytical solution of the radial Schroedinger equation is of high importance in non relativistic quantum mechanics, because the wave function contains all necessary information for full description of a quantum system. There are only few potentials for which the radial Schroedinger equation can be solved explicitly for all n and l states. Many methods were developed to solve the radial Schroedinger equation exactly for l = 0 within these potentials. The radial Schroedinger equation for the Woods-Saxon potential can not be solved exactly for $l \neq 0$. It is well known that the Woods-Saxon potential is one of the important short-range potentials in physics. Furthermore, this potential was applied to numerous problems, in nuclear and particle physics, atomic physics, condensed matter, and chemical physics.
An alternative method known as the Nikiforov-Uvarov (NU)[1] method was proposed for solving the Schroedinger equation. It would be interesting and important to solve the non relativistic radial Schroedinger equation for Woods-Saxon potential [2] for $l \neq 0$, since it has been extensively used to describe the bound and continuum states of the interacting systems. In this work, we solve the radial Schroedinger equation for the standard Woods-Saxon potential using NU method, and obtain the energy eigenvalues and corresponding eigenfunctions for l=0 states.

*Electronic address: aprakashmonteiro@gmail.com

Theory

In NU method the standard form of the Schroedinger equation after appropriate coordinate transformation is

$$u'' + \frac{\tilde{\tau}(z)}{\sigma(z)}u' + \frac{\tilde{\sigma}(z)}{\sigma^2(z)}u = 0 \qquad (1)$$

where $\sigma(z)$ and $\tilde{\sigma}(z)$ are polynomial of the second degree at most and $\tilde{\tau}(z)$ is a first degree polynomials.
To solve second order differential equations, the NU method can be used with appropriate coordinate transformation $z = z(r)$. The following equation is a general form of the Schroedinger like equation written for any potential

$$\left[\frac{d^2}{dz^2} + \frac{\alpha_1 - \alpha_2 z}{z(1-\alpha_3 z)}\frac{d}{dz} + \frac{-\xi_1 z^2 + \xi_2 z - \xi_3}{[z(1-\alpha_3 z)]^2}\right]u_n(z) = 0 \qquad (2)$$

According to the NU method, the eigenfunction and eigen energy condition become, respectively [3]

$$\begin{aligned} u(z) &= z^{\alpha_{12}}(1-\alpha_3 z)^{-\alpha_{12}-\frac{\alpha_{13}}{\alpha_3}} \\ & P_n^{(\alpha_{10}-1,\frac{\alpha_{11}}{\alpha_3}-\alpha_{10}-1)}(1-2\alpha_3 z) \end{aligned} \qquad (3)$$

$$\begin{aligned} &\alpha_2 n - (2n+1)\alpha_5 + (2n+1)\left(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}\right) \\ &+n(n-1)\alpha_3 + \alpha_7 + 2\alpha_3\alpha_8 + 2\sqrt{\alpha_8\alpha_9} = 0 \end{aligned} \qquad (4)$$

Where

$$\alpha_4 = \frac{1}{2}(1-\alpha_1), \quad \alpha_5 = \frac{1}{2}(\alpha_2 - 2\alpha_3)$$
$$\alpha_6 = \alpha_5^2 + \xi_1, \quad \alpha_7 = 2\alpha_4\alpha_5 - \xi_2$$
$$\alpha_8 = \alpha_4^2 + \xi_3, \quad \alpha_9 = \alpha_3\alpha_7 + \alpha_3^2\alpha_8 + \alpha_6$$

and

$$\alpha_{10} = \alpha_1 + 2\alpha_4 + 2\sqrt{\alpha_8}, \quad \alpha_{11} = \alpha_2 - 2\alpha_5 + 2\left(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}\right)$$
$$\alpha_{12} = \alpha_4 + \sqrt{\alpha_8}, \quad \alpha_{13} = \alpha_5 - \left(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}\right)$$

Results

We intend to solve Schroedinger equation for the Woods-Saxon (WS) potential which is given by

$$V(r) = -\frac{V_0}{1 + exp\left(\frac{r-R}{a}\right)} \tag{5}$$

The Schroedinger equation for the potential V(r) is given by

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right)\psi(r) = E\psi(r) \tag{6}$$

The radial part of above equation with WS potential is given by

$$R''(r) + \frac{2m}{\hbar^2}\left[E + \frac{V_0}{1 + qe^{2\alpha r}}\right]R(r) = 0 \tag{7}$$

where $\psi(r) = \frac{R(r)}{r}$ and $r - R_0 = r$, $\frac{1}{a} = 2\alpha$. We insert an arbitrary constant q within the potential for calculation purpose. By applying a transformation $z = -e^{2\alpha r}$ and $z^2 = e^{4\alpha r}$ to the above equation, we get

$$R'' + \frac{(1-qz)}{z(1-qz)}R' + \frac{1}{z^2(1-qz)^2} \times \left[-\epsilon q^2 z^2 + (2\epsilon q - q\beta)z + \beta - \epsilon\right]R = 0 \tag{8}$$

where

$$\epsilon = \frac{-mE}{2\hbar^2\alpha^2}, \beta = \frac{mV_0}{2\hbar^2\alpha^2} \tag{9}$$

From Eq. 8, we have $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = 1$
We chose $q = 1$.
$\xi_1 = \epsilon$, $\xi_2 = (2\epsilon - \beta)$, $\xi_3 = \epsilon - \beta$

$$\alpha_4 = 0, \quad \alpha_5 = \frac{-1}{2}, \quad \alpha_6 = \frac{1}{4} + \epsilon,$$
$$\alpha_7 = -(2\epsilon - \beta) \quad \alpha_8 = \epsilon - \beta, \quad \alpha_9 = \frac{1}{4} \tag{10}$$

$$\alpha_{10} = 1 + 2\sqrt{\epsilon - \beta}, \quad \alpha_{11} = 3 + 2\sqrt{\epsilon - \beta}$$
$$\alpha_{12} = \sqrt{\epsilon - \beta}, \quad \alpha_{13} = -1 + \sqrt{\epsilon - \beta} \tag{11}$$

Substituting Eq. 10 and Eq. 11 in Eq. 3 and Eq. 4 respectively, we obtain eigenfunction and eigen energy

$$R(z) = z^{\sqrt{\epsilon-\beta}}(1-z)P_n^{(2\sqrt{\epsilon-\beta},1)}(1-2z) \tag{12}$$

$$E = -\frac{\hbar^2}{2ma^2}\left[\left(\frac{mV_0a^2}{\hbar^2(n+1)}\right)^2 + \left(\frac{(n+1)}{2}\right)^2 + \frac{mV_0a^2}{\hbar^2}\right] \tag{13}$$

Conclusion

We have analytically deduced energy eigenvalues and corresponding eigenfunctions using the Woods-Saxon potential of the bound states. The Schroedinger equation is solved exactly for Woods-Saxon potential. Solutions are obtained reducing the Schroedinger equation into a second order differential equation by using an appropriate coordinate transformation. The Nikiforov-Uvarov method is used to obtain energy eigenvalues and the corresponding wave functions of the bound states, by fixing various parameters in question.

References

[1] A. F. Nikiforov and V. B. Uvarov, Special Functions of Mathematical Physics (Basel, Birkhauser, 1988).
[2] R. D. Woods and D. S. Saxon, Phys. Rev. **95**, 577 (1954).
[3] M. Hamzavi and M. Movahedi and K.-E. Thylwe and A. A. Rajabi, Chinese Phys. Lett. **29**, 080302 (2012).