Semiclassical triton

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Introduction

Three-body quantum mechanical systems occupy a special place in Physics. Triton is the simplest nucleus, consisting of a proton and two neutrons. It is bound by strong interaction. We obtain the ground state of triton semiclassically. This result is exact insofar as any $\hbar$-order correction can be obtained systematically.

The ground state of triton is an admixture of ${}^2S_{\frac{1}{2}}, {}^2P_{\frac{1}{2}}, {}^4P_{\frac{1}{2}}$ and ${}^4D_{\frac{1}{2}}$ states, consistent with $J^\pi = \frac{1}{2}$. However, ${}^2S_{\frac{1}{2}}$ and ${}^4D_{\frac{1}{2}}$ makes for 99.3% [6]. Here we begin with the coupled equations of the wavefunctions of the triton [1, 2], and then we treat it by semiclassical matrix method to obtain the S and D radial components (i.e. $u(s)$ and $w(s)$) [3].

Semiclassical Hamiltonian

The coupled second order differential equations of the radial components of S and D state wavefunctions $u(s)$ and $w(s)$ are given by [2]:

$$-\frac{\hbar^2}{m}\left(\frac{d^2u}{ds^2} - \frac{15u}{4s^2}\right) + \mathcal{V}^+(s)u + \tilde{\mathcal{V}}^t(s)w = \frac{14}{15}Eu. \qquad (1)$$

$$-\frac{\hbar^2}{m}\left(\frac{d^2w}{ds^2} - \frac{63w}{4s^2}\right) + [\mathcal{V}^{ct}(s) - \mathcal{V}^t(s)]w + \tilde{\mathcal{V}}^t(s)u = \frac{14}{15}Ew. \qquad (2)$$

where, $\mathcal{V}^+(s)$, $\tilde{\mathcal{V}}^t(s)$, $\mathcal{V}^{ct}(s)$, $\mathcal{V}^t(s)$, are as defined in [2] and $V^s(s)$, $V^t(s)$, $V^{ct}(s)$ are spin-singlet, tensor and central potentials respectively, which are parametrized using Yukawa type potentials.

We begin by writing (1) and (2) in a matrix form such that it could be written as an eigenvalue problem, $H\Psi = E\Psi$, with

$$\Psi = \begin{bmatrix} u \\ w \end{bmatrix}; \qquad \mathbf{H} = \left(-\frac{\hbar}{M}\frac{d^2}{ds^2}\right)\mathbf{I} + \mathbf{V}(s),$$

$$\mathbf{V}(s) = \begin{bmatrix} \frac{15\hbar^2}{4Ms^2} + \frac{15\mathcal{V}^+(s)}{14} & \frac{15\tilde{\mathcal{V}}^t(s)}{14} \\ \frac{15\tilde{\mathcal{V}}^t(s)}{14} & \frac{63\hbar^2}{4Ms^2} + \frac{15(\mathcal{V}^{ct}(s)-\mathcal{V}^t(s))}{14} \end{bmatrix}.$$

This can be diagonalised trivially to give the following potential-energy surfaces:

$$v_\pm(s) = \frac{1}{4}(\cos(\theta(s))\left(\frac{24\hbar^2}{Ms^2} \mp 2\mathcal{V}^{ct} \mp 2\mathcal{V}^+ \mp 2\mathcal{V}^t\right) + \frac{1}{4}\left(\frac{39\hbar^2}{Ms^2} \mp 4\tilde{\mathcal{V}}^t\sin(\theta(s)) + 2\mathcal{V}^{ct} + 2\mathcal{V}^+ - 2\mathcal{V}^t\right). \qquad (3)$$

with
$\tan\theta(s) = \left[\frac{10Ms^2\tilde{\mathcal{V}}^t(s)}{5Ms^2(\mathcal{V}^{ct}(s)-\mathcal{V}^+(s)-\mathcal{V}^t(s))+56\hbar^2}\right]$.
As we can see from Fig. 1, $v_+(s)$ is a binding

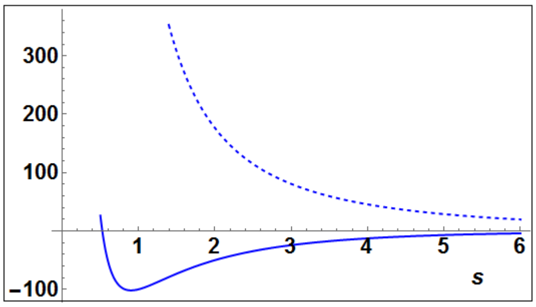

FIG. 1: $v_+(s)$ is a binding potential energy surface whereas $v_-(s)$ is a scattering surface.Here we have used the parameters of the Yukawa type potential from Reference [5].

potential energy surface and $v_-(s)$ is a scattering surface. They intersect in the complex plane.

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The diagonalised Hamiltonian can be written as:

$$h^{\pm} = \left(-\frac{\hbar^2}{M}\right)\left[1 \pm \frac{\hbar^2}{M(v_+(s)-v_-(s))}\left(\frac{d\theta(s)}{dr}\right)^2\right] \frac{d^2}{dr^2} + v_{\pm}(s) + \frac{\hbar^2}{8M}\left(\frac{d\theta(s)}{dr}\right)^2 \tag{4}$$

with semiclassical wave equations $h^{\pm}\psi^{\pm} = E\psi^{\pm}$; with $\psi^{+}(s) = u(s)\cos\frac{\theta(s)}{2} - w(s)\sin\frac{\theta(s)}{2}$ and $\psi^{-}(s) = w(s)\cos\frac{\theta(s)}{2} + u(s)\sin\frac{\theta(s)}{2}$ where $\theta(s)$ is a parameter which gives the mixing angle.

Ground State

In case of triton, the ground state occurs at -8.48 MeV. We follow the WKB quantisation for the radial problem as discussed in [3, 4] and we calculate $S/h = (n+1/2)$ (where $n = 0, 1, 2, \ldots$) over the classical turning points.

$$\frac{S}{h} = \frac{1}{h}\int_{0.54}^{4.61} ds\sqrt{M_+}\sqrt{-8.48 - v_+ - \frac{\hbar^2}{8M}\left(\frac{d\theta}{ds}\right)^2} \approx 0.488 \tag{5}$$

consistent with the ground state for $n = 0$, $M_+(s)$ is the effective mass

$$M_+(s) = \frac{M}{1 + \frac{\hbar^2}{M}\frac{1}{v_+(s)-v_-(s)}\left(\frac{d\theta(s)}{ds}\right)^2} \tag{6}$$

In order to evaluate $\psi^+(s)$, we use WKB method [4]. Expanding the potential, $U(s) = \frac{M}{\hbar^2}v_+(s) + \left(\frac{d\theta}{2ds}\right)^2$, in a Taylor series about the second turning point leads us to:

$$\frac{d^2\psi^+(s)}{ds^2} = \frac{M}{\hbar^2}(s-s_2)U'(s_2)\psi(s) \tag{7}$$

This is an Airy equation whose solutions are known. These solutions connect the oscillatory part in the well to the exponentially decaying part outside the well. The form for large negative and large positive y is as given:

$$\begin{aligned} Ai(y) &\approx |y|^{-1/4}\sin\left(\frac{2}{3}|y|^{3/2} + \frac{1}{4}\pi\right) \\ Ai(y) &\approx \frac{1}{2}y^{-1/4}exp\left(-\frac{2}{3}y^{3/2}\right) \end{aligned} \tag{8}$$

In this case, $y = \alpha s$ where, $\alpha = \left(\frac{M}{\hbar^2}U'(s-s_2)\right)^{\frac{1}{3}}$

By using the above relations of $\psi^+(s)$,$u(s)$ and $w(s)$ we can find $u(s)$ and $w(s)$. They are as plotted in Fig. 2, which remarkably resemble the plots in literature [5].

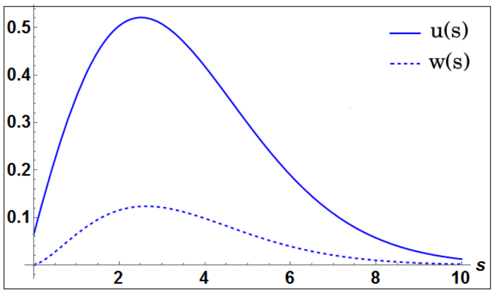

FIG. 2: Radial components of wavefunctions for S and D-states of triton.

Conclusion

The ground state wavefunctions of triton are studied using semiclassical treatment of coupled equations. The diagonalised potential matrix shows that the triton has a binding potential and a scattering potential surface. The system evolves on both the potential surfaces as they are connected at a complex point s_0 which can be found by $v_+(s) = v_-(s)$. By using the parametrisation of the Yukawa type potential, we see that the mixing angle θ weakly depends on s and remains small. The S and D radial wavefunctions are plotted which resemble numerical results seen in literature.

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