The α-decay half-life from resonance pole of S-matrix with exact explicit analytical solutions

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The process of decay of α particle from a radioactive nucleus is viewed as a quantum resonance state of a two-body scattering process of the α+daughter nucleus pair governed by a nucleus-nucleus potential. To match closely with this Coulomb+nuclear potential in the interior region, the following novel form of potential function [1]

$$V_{eff}(r) = H_0 \left\{ 1 - [\frac{1 - e^{-(r-R_0)/a}}{1 + ce^{-(r-R_0)/a}}]^2 \right\} \quad (1)$$

is used. Here, it has four parameters namely H_0, R_0, c and a. With the values of the radial position of the barrier obtained by using global formula $R_0=R_B=r_0(A_1^{1/3} + A_2^{1/3})$+2.72 fm with r_0=1.07 fm, the height of the barrier $H_0=V_B=\frac{Z_1 Z_2 e^2}{R_B}(1 - \frac{a_g}{R_B})$ MeV with a_g=0.6 fm along with the values of diffuseness a≈3 fm and constant c=0.44, the effective potential V_{eff}(r) (1) matches with the Coulomb+nuclear potential of a typical α+daughter nucleus system with mass number A_1=4 and proton number Z_1=2 of the projectile (α) and mass number A_2=208 and proton number Z_2=82 for the daughter nucleus ^{208}Pb.

The s-wave radial Schrödinger equation with the above potential (1) is solved exactly to give the solution as

$$u(r) = A\ z^{ir}(1 - z)^{-ir-is} F(l', m', n', \tfrac{1}{1-z}) + B\ z^{-ir}(1 - z)^{ir+is} F(l, m, n, \tfrac{1}{1-z}),$$

where z=−c $e^{-(r-R_0)/a}$ and F(l,m,n$\frac{1}{1-z}$) is the hyper-geometric function. The other terms are

$l = \frac{1}{2} - ir - it - is$, $m = \frac{1}{2} - ir + it - is$, $n = 1 - 2is$

$l' = \frac{1}{2} + ir - it + is$, $m' = \frac{1}{2} + ir - it + is$, $n' = 1 + 2is$,

r=ka, t=$[q^2(b+1)^2 - \frac{1}{4}]^{1/2}$, s=$[q^2(b^2-1) + r^2]^{1/2}$, k=$\sqrt{\frac{2\mu}{\hbar^2}E}$, $q^2 = \frac{2\mu}{\hbar^2} H_0 a^2$, b=$\frac{1}{c}$, reduced mass μ=$\frac{A_1 A_2}{A_1+A_2} m_n$ with m_n as mass of nucleon and E stands for the incident energy.

Using the boundary condition u(r=0)=0, we get $z_0 = -c\ e^{R_0/a}$ and

$$C_p = -\frac{B}{A} = z_0^{2ir}(1 - z_0)^{-2ir-2is} \frac{F(l',m',n',\frac{1}{1+ce^{R_0/a}})}{F(l,m,n,\frac{1}{1+ce^{R_0/a}})}.$$

As $ce^{R_0/a}$ >> 1, 1/(1+c$e^{R_0/a}$) ≃ 0, F(l,m,n,0)=1, we get C_p ≃ $e^{-2\pi r} c^{-2is} e^{-2iR_0 s/a}$.

The logarithmic derivative of the wave function at r=R_0 is given by

$$f(R_0) = \frac{du/dr}{u}\,|_{r=R_0} = (\frac{u'_-}{u_-}) \frac{1 - C_p(\frac{u_+}{u_-})(\frac{u'_+/u_+}{u'_-/u_-})}{1 - C_p(\frac{u_+}{u_-})}. \quad (2)$$

$$\frac{u'_-}{u_-} = \frac{c}{a}[-\frac{ir}{c} + \frac{ir+is}{1+c} + \frac{(l'm'/n')}{(1+c)^2} \frac{F(l'+1,m'+1,n'+1,\frac{1}{1+c})}{F(l',m',n',\frac{1}{1+c})}],$$

$$\frac{u'_+}{u_+} = \frac{c}{a}[\frac{ir}{c} - \frac{ir+is}{1+c} + \frac{(lm/n)}{(1+c)^2} \frac{F(l+1,m+1,n+1,\frac{1}{1+c})}{F(l,m,n,\frac{1}{1+c})}],$$

$$\frac{u_+}{u_-} = exp(2\pi r)\ c^{-2ir}(1 + c)^{2(ir+is)} \frac{F(l,m,n,\frac{1}{1+c})}{F(l',m',n',\frac{1}{1+c})},$$

In the region $r > R_0 = R_B$ where the potential is pure Coulombic, the Coulomb wave functions (regular F_0 and irregular G_0) and their derivatives (F'_0 and G'_0) with respect to $\rho = kr$ for $\ell = 0$ case are expressed [2] as

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$F_0 = \frac{1}{2}\beta\ exp(\alpha)$, $F_0' = (\beta^{-2} + \frac{1}{8\eta}\gamma^{-2}\beta^4)F_0$, $G_0 = \frac{\beta}{exp(\alpha)}$, $G_0' = [-\beta^{-2} + \frac{1}{8\eta}\gamma^{-2}\beta^4]G_0$, $\gamma = \frac{\rho_0}{2\eta}$, $\beta = [\frac{\gamma}{1-\gamma}]^{\frac{1}{4}}$, $\alpha = 2\eta([\gamma(1-\gamma)]^{\frac{1}{2}} + arcsin\ \gamma^{\frac{1}{2}} - \frac{1}{2}\pi)$, $\rho_0 = kR_0$, $\eta = \frac{\mu}{\hbar^2}\frac{Z_1Z_2e^2}{k}$. By requirement of continuity at r=R_0, the wave functions and their derivatives are matched at r=R_0 to obtain the S-matrix denoted by S(k) as

$$S(k) = \frac{2ikF_0' - 2iF_0 f(R_0) + f(R_0)[G_0 + iF_0] - k[G_0' + iF_0']}{f(R_0)[G_0 + iF_0] - k[G_0' + iF_0']}, \quad (3)$$

where $f(R_0)$ is given by (2).

From the variation of the phase-shift δ(k) of the S-matrix, S(k)=$e^{2i\delta}$, as a function of k we locate the pole of S(k) that manifests the resonance state at the energy equal to the Q-value of decay. The diffuseness parameter 'a' of the potential (1) is varied to obtain the above situation of resonance with pole k=$k_r - ik_i$ at the required Q-value. From this pole, we derive the decay half-life $T_{1/2} = \frac{0.693\mu}{2\hbar k_r k_i}$. We denote this result of half-life by $T_{1/2}^{(anal)}$ as it is based on exact analytical solutions. We apply the formulation to the process of α-decay of the ^{212}Po nucleus represented by the resonance of the two-body, α+^{208}Pb, system. The effective potential (1) with parameters $r_0 = 1.07$fm, $a_g = 0.6$fm, c=0.44 and a=3.17 fm is found to generate the resonance at the required energy Q_α=8.954 MeV giving decay half-life $T_{1/2}^{(anal)}$ =2.98x10^{-7}s which is very close to the experimental result $T_{1/2}^{(expt)}$=2.99x10^{-7}s. As the potential (1) uses global formula for its parameters for barrier position and height, one can apply this formulation of estimation of decay rate to other α+daughter nucleus pairs with little variation of the the value of diffuseness parameter 'a' around 3 fm to generate the proper resonances of the systems and obtain the results of decay half-lives from the poles describing these resonances. We calculate the results of $T_{1/2}^{(anal)}$ for several α-emitters including heavy and super-heavy nuclei and compare them with the corresponding experimental values denoted by $T_{1/2}^{(expt)}$ in Table 1. Our calculated $T_{1/2}^{(anal)}$ results account for the experimental $T_{1/2}^{(expt)}$ data in the respective nuclei quite well. It may be pointed out here that the $T_{1/2}^{(expt)}$ data ranging from very small value (10^{-7}s) to large value of the order of 10^{17}s in different α-emitters are accounted for by our calculation with close proximity.

TABLE I: Comparison between the experimental α-decay half-lives and calculated half-lives of different α-emitters.

Parent	$Q_\alpha^{(expt)}$(MeV)	$T_{1/2}^{(expt)}$(s)	$T_{1/2}^{(anal)}$(s)
^{212}Po	8.954	2.99x10^{-7}	2.98x10^{-7}
^{214}Rn	9.208	2.70x10^{-7}	1.26x10^{-7}
^{224}Ra	5.789	3.33x10^5	4.47x10^5
^{232}Th	4.082	5.69x10^{17}	12.7x10^{17}
^{236}U	4.573	1.00x10^{15}	1.46x10^{15}
^{244}Pu	4.666	3.17x10^{15}	2.89x10^{15}
^{248}Cm	5.162	1.43x10^{13}	0.89x10^{13}
^{252}Cf	6.214	1.02x10^8	1.49x10^8
^{250}Fm	7.557	2.28x10^3	1.23x10^3

References

[1] T Tietz, *J. Chem. Phys.* **38**, 3036 (1963).
[2] M Abramowitz, I A Stegun, *Handbook of Mathematical Function* (Dover, New York, 1965) p. 542.