A Green's Function Approach to the Non Relativistic Wave Equation with Hulthen Potential

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Introduction

The Hulthen potential is one of the important short-range potential. Which is at short distances possesses asymptotic freedom. It behaves like the Coulomb potential for small values of r and decreases exponentially at large values. This behavior is in particular of interest in particle Physics.[3]. Moreover, the potential has been used in other areas such as nuclear, atomic, solid-state, and chemical physics. The Hulthen potential has received extensive study in both relativistic and non relativistic quantum mechanics. The Hulthen potential behaves like a short range potential and we can use it for the perturbation part. The Schrodinger wave equation is one of the fundamental wave equation. Its solution play an essential part in particle Physics.There are various methods available in literature for its solution. The Green's function technique takes a different approach, by excluding a part of the potential term from the solution initially which can be included later after finding the Green's function.[1]

Theoretical Background

The Hulthen potential V_H is defined as the

$$V_H(r) = -Ze^2\mu\frac{\exp(\frac{-r}{\mu})}{1-\exp(\frac{-r}{\mu})} \tag{1}$$

The radial part of Schrondinger wave equation for the relative motion of two particles interacting via Hulthen potential can be written as

$$\Big[-\frac{\hbar^2}{2\mu}\frac{1}{r^2}\frac{d}{dr}\big(r^2\frac{d}{dr}\big)+V_{eff}(r)\Big]u(r) = Eu(r) \tag{2}$$

Making the standard change $R(r) = r^{-1}u(r)$.

$$\Big[-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2}+V_{eff}(r)\Big]u(r) = Eu(r) \tag{3}$$

The $V_{eff}(r)$ is the effective potential, which consists of generalized Hulthen potential $V_H(r)$ and the centrifugal term. The equation 2 can not be solved analytically due to the centrifugal term, we have to use a proper approximation of this term.The equation 2 allow us to obtain,[2]

$$\Big[-\frac{\hbar^2}{2\mu}\frac{d}{dx^2}+V_{eff}(x)\Big]u(x) = Eu(x) \tag{4}$$

Where

$$V_{eff}(r) = -Zq^2\delta\frac{e^{-x}}{1-e^{-x}}+ \frac{l(l+1)\hbar^2\delta^2}{2\mu}\Big[c_0+\frac{c_1e^{-x}}{(1-e^{-x})}+\frac{c_2e^{-2x}}{(1-e^{-x})^2} \tag{5}$$

Where $\delta r = x$, and substituting 5 in equation 4 and equation 4 is integrable under the exchange of variables $z = e^{-x}$. After introducing this new variable we can rearrange the equation 4 as

$$\frac{d^2u(z)}{dz^2}+\frac{1}{z}\frac{du(z)}{dz}-\Big[\frac{l(l+1)c_2z^2+c_2}{z^2(1-z)^2}+ \frac{\lambda^2}{z^2}\Big]u(z) = -\frac{\alpha}{z(1-z)}u(z) \tag{6}$$

Where we used the dimensionless parameters given by [2]

$$\lambda^2 = -\frac{2\mu E}{\hbar^2\delta^2}+l(l+1) \quad \alpha = \frac{2\mu Zq^2}{\hbar^2\delta}-l(l+1)c_1$$

satisfying the boundary condition at z=0 and z=1, In terms of Green's function $g(z,z')$. the solution of Equation 6 will be obtained by solving the integral equation.

$$u(z) = \alpha\int_0^1 g(z,z')u(z')\frac{1}{z'(1-z')}dz' \tag{7}$$

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By definition of Greens function

$$\Big[\frac{d^2}{dz^2}+\frac{1}{z}\frac{d}{dz}-\frac{l(l+1)c_2z^2+c_2}{(1-z)^2}+ \lambda^2\Big]g(z,z')=-\delta(z-z')\frac{1}{z'(1-z')} \quad (8)$$

Under the boundary condition $\lim_{z\to 0} z(z-1)g(z-z')=0$, equation 6 easily handled using the following supposition

$$u(z,z')=z^{\lambda}(1-z)^{\beta}g(z,z') \quad (9)$$

By substitution equation 9 we get the following second order homogeneous differential equation

$$z(z-1)\frac{d^2g(z,z')}{dz^2}+\Big[-(2\lambda+2\beta+1)z+ 2\lambda+1\Big]\frac{dg(z,z')}{dz}+\Big[\alpha-(2\lambda+1)\beta\Big]g(z,z')=0 \quad (10)$$

only when we make the following choice for

$$\beta=\frac{1}{2}\sqrt{1+4l(l+1)c_2}+\frac{1}{2} \quad (11)$$

The equation 10 corresponds to the well known hypergeometric equation.

$$z(z-1)\frac{d^2f(z)}{dz^2}+ \Big[-(a+b+1)z+c\Big]\frac{df(z)}{dz}-abf(z)=0 \quad (12)$$

Whose solution is the generalized hypergeometric function

$$g(z,z')=C_1.{}_2F_1(a,b,c,z)+C_2z^{(1-c)} F_1(a-c+1,b-c+1;2-c;z) \quad (13)$$

comparing equation 10 and 12

$$a=\lambda+\beta-\sqrt{\alpha+\lambda^2+\beta^2-\beta}$$
$$b=\lambda+\beta+\sqrt{\alpha+\lambda^2+\beta^2-\beta}$$
$$c=1+2\lambda$$

considering the boundary condition that is g(z,z') tending to to finite value when $z\to 0$ the allowed solution is [2]

$$g(z,z')=C_1.{}_2F_1(a,b,c;z)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!} \quad (14)$$

where $(a)_k=\frac{\Gamma(a+k)}{\Gamma(a)}$ denotes the Pochhammer symbol. Using $\lambda+\beta-\sqrt{\alpha+\lambda^2+\beta^2-\beta}=n_r$ and $b=2\lambda+2\beta+n_r$, The general solution of equation 6

$$u_{n,l}(z)=N_{n,l}z^{\lambda}(1-z)^{\beta}. {}_2F_1(-n_r,2\lambda+2\beta+n_r;1+2\lambda;z) \quad (15)$$

Where N_{nl} is the normalization constant. This constant can be calculated from the normalizing condition.

Radial wave equation

The radial wave function can be found by integrating equation 7 iteratively. As u(z) also appear on the right-hand side of equation 7, therefore, we propose its value under the restriction that it must be finite everywhere from z=0 and z=1. According to 6 the radial wavefunction has the form ${z'}^{\lambda+1}$ nearly z equals to zero and has the form of $(1-z)^{\beta+1}$ near z=1. Therefore in the present case an appropriate choice for the approximate value of u(z') will be

$$u(z')={z'}^{\lambda+1}(1-z')^{\beta+1} \quad (16)$$

With equation 16 the wave function remains finite in the range 0 to 1. However, the exact solution can be found by solving equation 7 iteratively. The proposed wave function equation and Greens function g(z,z') are inserted into equation 7 and the integration is carried out to obtain the solution u(z). In the next iteration the resultant wave u(z) is considered as a proposed wave function to get another solution. The process is repeated several time for a better approximation to the standard solution given by equation 15.

References

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