Validity of Hill-Wheelar and WKB transmission coefficients at astrophysical energies

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Introduction

The study of fusion cross-section at deep sub-barrier energy regions for reactions involving light nuclei is of substantial importance in probing various astrophysical phenomena. In particular, the ^{12}C+^{12}C, ^{12}C+^{16}O and ^{16}O+^{16}O reactions play a central role in the later evolution stages of massive stars [1, 2]. The temperatures at which these reactions proceed in the stellar environments correspond to the centre of mass energies which lie far below the Coulomb barrier. As the experimental measurements for these reactions at stellar energies are extremely strenuous, so reliable theoretical predictions become essential [1, 2].

The below-barrier nuclear fusion can only be explained through the quantum mechanical tunnelling of the barrier formed due to the interaction of nuclear, Coulomb and centrifugal potentials. Out of different methods developed to calculate the barrier transmission probability, the Hill-Wheeler (HW) [3] and Wentzel-Kramers-Brillouin (WKB) [4] approaches have been widely adopted in nuclear physics. The HW method is based on the parabolic barrier approximation whereas the semi-classical WKB approximation uses the exact barrier shape. The characteristics of the fusion barrier such as its shape, height and frequency have a direct impact on the fusion cross-section at below barrier energies. Thus it is crucial to assess the validity of HW and WKB approximations which use different shapes of fusion barriers to determine the transmission probability in describing the fusion dynamics of ^{12}C+^{12}C reaction at astrophysically significant energies. The prerequisite to study the fusion mechanism is the evaluation of nuclear potential formed between the two interacting nuclei. In the present analysis, the nuclear potential is calculated using the double folding model embellished with the nuclear densities and R3Y effective nucleon-nucleon interaction potential obtained within the relativistic mean-field formalism [3].

Theoretical Formalism

The characteristics of the fusion barrier obtained using RMF densities and the R3Y effective NN potential [3] are used to obtain the penetration probability within the HW and WKB approximation. The HW transmission coefficient is evaluated using the parabolic barrier approximation [3] and is written as,

$$P_\ell^{HW} = \left[1 + exp\left(\frac{2\pi(V_B^\ell - E_{c.m.})}{\hbar\omega_\ell}\right)\right]^{-1}. \quad (1)$$

The results of the HW approach are also compared with the WKB transmission coefficient [4] written as,

$$P_\ell^{WKB} = \exp\left[-\frac{2}{\hbar}\int_{r_1}^{r_2}[2\mu\{V_T^\ell(R) - E_{c.m.}\}]^{\frac{1}{2}}dR\right]. \quad (2)$$

The classic turning points i.e., r_1 and r_2 are the solutions of the equation $V_T^\ell(R) = E_{c.m.}$. These turning points merge at $E_{c.m.} = V_B$ and there is no real solution this equation for $E_{c.m.} > V_B$. Since the energies of astrophysical significance for ^{12}C+^{12}C reaction lie far below the fusion barrier [1, 2], so the results for WKB transmission coefficient are only obtained at sub-barrier energies. The cross-section is calculated within the $\ell-$summed Wong model [3] and is then rescaled in terms

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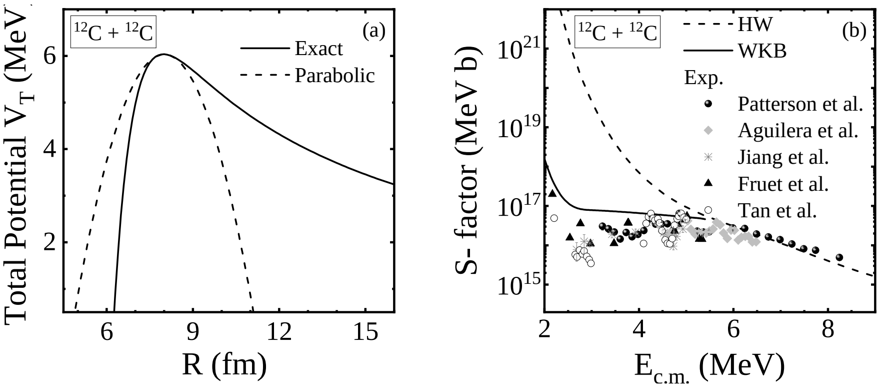

FIG. 1: (a)Fit to the exact fusion barrier at $\ell = 0\hbar$ (solid line) by pure parabolic (dashed line). (b) The astrophysical S-factor (MeV b) for $^{12}C+^{12}C$ reaction calculated using the WKB (solid line) and HW (dashed line) transmission coefficients. The experimental data [1, 2] are also given for comparison. See text for details.

of the astrophysical S-factor [1, 2] to reduce the Coulomb penetration effects.

Results and Discussion

As mentioned above, in the Hill-Wheeler approach the analytic expression for transmission coefficient is obtained under the assumption of a pure parabolic fusion barrier. The semi-classical WKB approximation on the other hand uses the exact barrier shape. The barrier region of the exact interaction potential (solid line) obtained within the RMF formalism and the fusion barrier obtained using the parabolic function (dashed line) with parameters fitted to the exact barrier are shown in Fig. 1(a) as a function of radial separation distance (R) for s-wave ($\ell = 0$). It can be observed from fig. 1(a) that for $^{12}C+^{12}C$ reaction, the difference between the shape of the parabolic and exact barrier becomes more prominent at the larger separation distance. Moreover, unlike the parabolic barrier, the plot of the exact barrier is asymmetric around the barrier position (R_B).

To assess, the validity of the HW approach which uses the parabolic barrier and the WKB approach which uses the exact barrier shape at stellar energies, the calculated astrophysical S-factor for $^{12}C+^{12}C$ reaction is compared. Fig. 1(b) shows the astrophysical S-factor (MeV b) obtained using the WKB (solid) and HW (dashed line) transmission coefficients as a function of center of mass energies $E_{c.m.}$ (MeV). The experimental data from [1, 2] are also given for comparison. It can be observed from Fig. 1(b) that the HW approximation of parabolic barrier gives a reasonable match to experimental data at the above and around the barrier energies but overestimates the S-factor at far below barrier energies. However, the WKB transmission coefficient using the exact barrier shape of the interaction potential obtained from RMF formalism gives a better match to the experimental data in this energy region. All these observations infer that the shape of the fusion barrier at a larger separation distance has a significant influence on the S-factor of $^{12}C+^{12}C$ reaction at stellar energies. Consequently, the WKB approximation embellished with RMF formalism gives more reliable results at the stellar energies than the HW approximation of the parabolic barrier. However, these results are preliminary and a more comprehensive investigation involving more reactions of astrophysical significance is under process.

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