On-shell Jost function for the Deng-Fan potential in all partial waves

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Introduction

The Deng-Fan potential, a multiparameter exponential type potential, is commonly used to characterize the molecular vibrational spectrum. Given its relevance to the nuclear world, the Deng-Fan potential is already an appealing alternative for theoretical physicists. One of the main features of the Deng-Fan potential is its ability to accurately reproduce experimental scattering data. This ability is especially significant in low-energy regimes, where the interaction of forces is extremely sensitive to potential parameters. Several articles have explored the approximate analytical solutions to numerous other exponential-type potentials. Saha et al. [1] derived the s-wave Jost function for the Deng-Fan potential in the context of low energy elastic nuclear scattering. Motivated from Ref. [1], we present the all partial wave on-shell Jost solution for the aforementioned potential using a differential equation technique. Near the origin behaviour of Jost solution yields the Jost function. Considering both the nature of the Deng-Fan potential and the screened centrifugal barrier we modify the basic definition in order to derive the all partial wave Jost function.

Methodology

The all-partial wave Deng-Fan potential [2] can be written as

$$V_{DF}(r) = V_1 \frac{e^{-\alpha r}}{1-e^{-\alpha r}} + \left(V_2 + \alpha^2 \ell(\ell+1)\right) \frac{e^{-2\alpha r}}{\left(1-e^{-\alpha r}\right)^2} \tag{1}$$

In the centre of mass system, the on-shell Deng-Fan Jost solution $f_{\ell DF}(k,r)$ satisfies the radial Schrödinger equation

$$\left\{ \frac{d^2}{dr^2} + k^2 - V_{DF}(r) \right\} f_{\ell DF}(k,r) = 0 \tag{2}$$

Changing the independent variable $e^{-\alpha r} = z$ and use of the following transformation

$$f_{\ell DF}(k,z) = z^{\rho}(1-z)^{\beta} F_{\ell}(k,z) \tag{3}$$

in Eq. (2), the resulting equation resembles the Gaussian hypergeometric differential equation

$$z(1-z)y''(z) + \left\{ c' - (a' + b' + 1)z \right\} y'(z) - a'b' y(z) = 0 \tag{4}$$

in the condition

$$\rho = \pm \frac{ik}{\alpha} \tag{5}$$

and

$$\beta = \frac{1}{2} \pm \frac{\sqrt{\alpha^2 + 4(V_2 + \alpha^2 \ell(\ell+1))}}{2\alpha}. \tag{6}$$

Choosing $\rho = -\frac{ik}{\alpha}$ and $\beta = \frac{1}{2} - \frac{\sqrt{\alpha^2 + 4(V_2 + \alpha^2 \ell(\ell+1))}}{2\alpha}$, the on-shell Jost solution is found as

$$f_{\ell DF}(k,r) = e^{ikr}\left(1-e^{-\alpha r}\right)^{\frac{1}{2} - \frac{\sqrt{\alpha^2+4(V_2+\alpha^2\ell(\ell+1))}}{2\alpha}} {}_2F_1\left(a', b'; 1 - \frac{2ik}{\alpha}; e^{-\alpha r} \right) \tag{7}$$

where

$$a' = \frac{1}{2} - \frac{\sqrt{\alpha^2 + 4(V_2 + \alpha^2 \ell(\ell+1))}}{2\alpha} - \frac{ik}{\alpha} \pm \frac{1}{\alpha}\left\{ -k^2 + (V_2 + \alpha^2 \ell(\ell+1)) - V_1 \right\}^{1/2} \tag{8}$$

$$b' = \frac{1}{2} - \frac{\sqrt{\alpha^2 + 4(V_2 + \alpha^2 \ell(\ell+1))}}{2\alpha} - \frac{ik}{\alpha} \mp \frac{1}{\alpha}\left\{ -k^2 + (V_2 + \alpha^2 \ell(\ell+1)) - V_1 \right\}^{1/2} .\tag{9}$$

It is well established that near the origin behaviour of the Jost solution represents the Jost function. To that end, for the Deng-Fan potential the basic definition is modified to have

$$f_{\ell DF}(k) = \lim_{r\to 0} \left(\frac{\sqrt{\alpha^2 + 4(V_2 + \alpha^2 \ell(\ell+1))}}{\alpha} \right) r^{\left(-\frac{1}{2} + \frac{\sqrt{\alpha^2+4(V_2+\alpha^2\ell(\ell+1))}}{2\alpha} \right)} f_{\ell DF}(k,r) .\tag{10}$$

Substituting Eq. (7) along with the following standard integral

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$$ {}_2F_1(d_1,d_2;d_3;1) = \frac{\Gamma(d_3)\Gamma(d_3-d_1-d_2)}{\Gamma(d_3-d_1)\Gamma(d_3-d_2)} \tag{11} $$

in Eq. (10) yields

$$ f_{\ell DF}(k) = \alpha^{\left(\frac{1}{2}-\frac{\sqrt{\alpha^2+4(V_2+\alpha^2\ell(\ell+1))}}{2\alpha}\right)} \frac{\Gamma\left(1-\frac{2ik}{\alpha}\right)\Gamma\left(1+\frac{\sqrt{\alpha^2+4(V_2+\alpha^2\ell(\ell+1))}}{\alpha}\right)}{\Gamma(a'')\Gamma(b'')} \quad ,(12) $$

where

$$ a'' = \frac{1}{2} + \frac{\sqrt{\alpha^2+4(V_2+\alpha^2\ell(\ell+1))}}{2\alpha} - \frac{ik}{\alpha} \pm \frac{1}{\alpha}\left\{-k^2+(V_2+\alpha^2\ell(\ell+1))-V_1\right\}^{1/2} \tag{13} $$

and

$$ b'' = \frac{1}{2} + \frac{\sqrt{\alpha^2+4(V_2+\alpha^2\ell(\ell+1))}}{2\alpha} - \frac{ik}{\alpha} \mp \frac{1}{\alpha}\left\{-k^2+(V_2+\alpha^2\ell(\ell+1))-V_1\right\}^{1/2} \quad .(14) $$

Equation (12) represents the on-shell Jost function for the Deng-Fan potential and negative of phase of the Jost function yields the scattering phase shift.

Results and Discussion

We have derived the on-shell Jost function for all partial waves for the Deng-Fan potential. As $V_2 \to 0$, the Deng-Fan potential becomes the Hulthén potential. We have verified that under the Hulthén limit, Eq. (12) corresponds to the Hulthén all partial wave Jost function [3], ensuring the accuracy of our built expression. Just to check the usefulness of our constructed expression, we exploit Eq. (12) to parameterize the nuclear Deng-Fan potential in order to generate the standard scattering phase shift data of n-p system for ℓ=0. For this purpose, we have used the exact values of $\hbar^2/2\mu = 41.47\ MeVfm^2$. The best fitted parameters of the Deng-Fan potential for the concerned system are enlisted in Table 1.

Table 1: Potential parameters for n-p system

States	V_1(fm^{-2})	V_2(fm^{-2})	α (fm^{-1})
$^1S_0$	-29.25	113.75	2.52
$^3S_1$	-31.1059	94	2.376

Table 2: (n-p) phase shift for different lab energies along with standard data [4].

E_{Lab} (MeV)	$^1S_0$ Phase shift (degree)	$^3S_1$ Phase shift (degree)
1	62.36 (62.105)	147.7 (147.748)
5	64.77 (63.689)	117.95(118.169)
10	61.22 (60.038)	102.16(102.587)
25	51.71 (51.011)	79.7 (80.559)
50	40.55 (40.644)	61.46 (62.645)
100	25.92 (26.772)	41.91 (43.088)
150	15.88 (16.791)	29.803 (30.644)
200	8.17 (8.759)	20.91 (21.24)
250	1.87 (1.982)	13.84 (13.55)
300	-3.45 (-3.855)	7.95 (6.966)
350	-8.09 (-8.923)	2.88 (1.176)

Table 2 shows that the scattering phase shifts of both the $^1S_0$ and $^3S_1$ states of the (n-p) system are consistent with those of Perez et al. [4]. The data from Perez et al. [4] are shown in brackets in Table 2. Our $^1S_0$ phase shift is able to change the sign at E_{Lab}=267 MeV and is in good agreement with [4]. As the zeros of the Jost function in the upper half of the complex plane represent the bound state, we have confirmed that the potential parameters for the $^3S_1$ state can generate the exact deuteron binding energy 2.225MeV. Moreover the $^3S_1$ phase shift observations are also consistent with the data of Perez et al. [4]. It is also verified that the phase shift data of all other states for ℓ=1 matches well with Ref. [4]. Thus the Jost function may be useful for investigating both the bound and scattering state problems.

References

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