Analytical solution of the radial Schrödinger equation with Approximated Varshni-Hellmann Potential

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Introduction

The aim is to find the Energy eigen values of the potential by solving N-dimensional Schrodinger equation. There are different analytical technique used to obtain solutions of wave equation.

Here we are using Nikiforov-Uvarov functional analytical method (NUFA) [2]. NUFA method is regarded as a combination of Nikiforov Uvarov (NU) [1] and the parametric (NU) and the functional analysis methods. Here the potential is approximated Varshni-Hellmann potential. Varshni potential is a short-range repulsive potential energy function that is important in chemical and molecular physics. The Varshni potential has been used to predict mass spectra of charmonium and bottomonium multiplets. The Hellmann potential is the combination of Coulomb and Yukawa potentials.

Theoretical Background

The radial part of the Schrödinger equation compared with NUFA method. In this we solve Schrödinger equation by using Varshni-Hellmann potential [4].

Hellmann potential is of the following form

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$$\boldsymbol{V(r) = \frac{-c}{r} + \frac{d}{r}e^{-\alpha r}}$$

Varshni potential is of the following form

$$\boldsymbol{V(r) = a + \frac{ab}{r}e^{-\alpha r}}$$

where r is the inter nuclear interval and c and d stand for the strong points of Coulomb and Yukawa potentials. where a and b denote the strong points of Varshni potential. The combined potential is

$$\mathbf{V(r)=a+\frac{d-ab}{r} + (ab-d)\alpha - \frac{(ab-d)\alpha^2 r}{2} + \frac{(ab-d)\alpha^3 r^2}{6} - \frac{c}{r}}$$

a, b are the strong points of Varshni potential is the inter nuclear interval and c, d are the strong points of the Coulomb and Yukawa potential.
Final form of the radial Schrödinger equation is NUFA method is of the general form.
The N-dimensional radial Schrödinger equation for CPP is written as

$$\boldsymbol{\frac{d^2R(r)}{dr^2} + \frac{(N-1)}{r}\frac{d}{dr} + \frac{2\mu}{\hbar^2}\left[E - V(r) - \frac{l(l+N-2)\hbar^2}{2\mu r^2}\right]}$$
$$\mathbf{R(r)=0}$$

$$\boldsymbol{\frac{d^2R(r)}{dr^2} + \frac{(N-1)}{r}\frac{d}{dr} + \frac{2\mu}{\hbar^2}\left[E - Ar^2 - Br^2 + \frac{C}{r} - \frac{l(l+N-2)\hbar^2}{2\mu r^2}\right]R(r) = 0}$$

$$\boldsymbol{\frac{d^2\psi(s)}{ds^2} + \frac{\alpha_1 - \alpha_2}{s(1-\alpha_3 s)}\frac{d\psi(s)}{ds} + \frac{1}{s^2(1-\alpha_3)^2}[-\xi_1 s^2 + \xi_2 s - \xi_3]\psi(s) = 0}$$

Now the approximated potential is

$$\boldsymbol{V(r) = -\frac{(ab-d)\alpha^2 r}{2} + \frac{(ab-d)\alpha^3 r^2}{6} + \frac{d-ab-c}{r}}$$

where we are equating to a Columbic potential $V(r) = Ar^2 + Br - \frac{D}{r}$
here

$A = \frac{(ab-d)\alpha^3}{6}$, $B = -\frac{(ab-d)\alpha^2}{2}$, $D = \frac{d-ab-c}{r}$

comparing these two above equation and substituting Varshni-Hellmann potential into it

we can find Energy eigen value and Radial part of
the wave function .
We are getting polynomials in hyper-geometric equation .These are

$\tau_1 = \frac{-\mu(ab-d)}{\hbar^2\alpha}$ $\tau_2 = \frac{\mu}{\hbar^2\alpha}(-\frac{11}{3}(ab-d)-2c)$
$\tau_3 = \frac{\mu}{\hbar^2\alpha}(-\frac{8}{3}(ab-d)-2c)+ l(l+N-2)$
General energy equation for the approximated potential is

$$E = \frac{-\hbar^2\alpha^2}{2} \left[\frac{-(\nu+n)^2+\tau_1-\tau_3}{2(\nu+n)}\right]^2 + \frac{\hbar^2\alpha^2\tau_3}{2}$$

The General form of radial part of the wave function given by

$R(r)$
$= N_1(e^{-\alpha r})^{\sqrt{\varepsilon+\tau_3}}(1-e^{-\alpha r})^{\frac{1}{2}+\frac{1}{2}\sqrt{1+4(\tau_1-\tau_2+\tau_3)}}$
$\times {}_2F_1(a_2, b_2, c_2;\ e^{-\alpha r})$
$= N_1(e^{-\alpha r})^{\left(\frac{-(\nu+n)^2+\tau_1-\tau_3}{2(\nu+n)}\right)}(1-$
$e^{-\alpha r})^{\frac{1}{2}+\frac{1}{2}\sqrt{1+4(l(l+N-2)}} \times {}_2F_1(a_2, b_2, c_2;\ e^{-\alpha r})$

Where N_1 is the normalization constant and ${}_2F_1(a_2, b_2, c_2; e^{-\alpha r})$ is hypergeometric function. Here the parameters a_2 , b_2 and c_2 are defined as

$a_2 = (\lambda + \nu + \sqrt{\xi_1})$
$b_2 = (\lambda + \nu - \sqrt{\xi_1})$ --------(13)

$$c_2 = 1 + 2\lambda$$

Conclusion

We analytically find Energy eigen values using the approximated Varshni-Hellmann potential of the bound state. The solution are obtained by reducing Schrödinger into a second-order differential equation. The Nikiforov-Uvarov method is used to obtain Energy eigen values. The results of the present work can be used to evaluate mass spectra of hadrons using the relation $M = (\sum_i m_i) + E$ where m_i gives the quark masses [3].

State	**α**	**E (eV)**
1s	0.025	-6.684
	0.050	-6.793
	0.075	-6.905
2s	0.025	-3.396
	0.050	-3.463
	0.075	-3.536
2p	0.025	-1.106
	0.050	-1.203
	0.075	-1.308
3s	0.025	-1.867
	0.050	-1.912
	0.075	-1.970
3p	0.025	-0.598
	0.050	-0.684
	0.075	-0.787
3d	0.025	-0.075
	0.050	-0.181
	0.075	-0.306

Table 1: Energy eigenvalues for different states

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