Second-order spin hydrodynamics from Zubarev's nonequilibrium statistical operator formalism

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Introduction

The measurement of Global Λ hyperon polarization in nuclear collisions [1] creates a new interest in studying the spin-dependent transport coefficients. On the other hand, with relativistic hydrodynamics being the most robust theory that explains the evolution of QGP, a theoretical interest is also created to include spin in relativistic hydrodynamics, namely relativistic spin hydrodynamics.

In this proceeding, we give an overview to derive the second-order dissipative tensors in relativistic spin hydrodynamics, *viz.* rotational stress tensor ($\tau_{\mu\nu}$), boost heat vector (q_μ), shear stress tensor ($\pi_{\mu\nu}$) and bulk viscous pressure (Π) by using the Zubarev's nonequilibrium statistical operator (NESO) formalism, which conjunction with the first-order results gives the evolution equation for the respective tensors.

This method starts by defining a NESO $\hat{\rho}(t)$ for a system that is in a hydrodynamic regime, containing spin degrees of freedom [2], given by

$$\hat{\rho}(t) = Q^{-1}\exp\Big[-\int d^3x \hat{Z}(\vec{x},t)\Big],$$

$$\text{where, } \hat{Z}(\vec{x},t) = \epsilon\int_{-\infty}^{t} dt_1 e^{\epsilon(t_1-t)}\Big[\beta^\nu(\vec{x},t_1)\hat{T}_{0\nu}(\vec{x},t_1) - \alpha(\vec{x},t_1)\hat{N}^0(\vec{x},t_1) - \frac{1}{2}\Omega_{\alpha\beta}(\vec{x},t_1)\hat{\Sigma}^{0\alpha\beta}(\vec{x},t_1)\Big],$$

here, ϵ is the small infinitesimal parameter, $Q = \mathrm{Tr}\exp\big[-\int d^3x\hat{Z}(\vec{x},t)\big]$, $\beta^\mu = \beta u^\mu$, $\alpha = \beta\mu$ and $\Omega_{\alpha\beta} = \beta\omega_{\alpha\beta}$, where β being the inverse of temperature, u^μ is the fluid four-velocity, μ is the particle chemical potential and $\omega_{\alpha\beta}$ is the spin chemical potential. The energy-momentum tensor $\hat{T}^{\mu\nu}$, particle current $\hat{N}^\mu$ and the spin angular momentum $\hat{\Sigma}^{\mu\alpha\beta}$ are operator-valued function. The $\hat{\rho}(t)$ can be decomposed into equilibrium and nonequilibrium parts by integrating $\hat{Z}(\vec{x},t)$ by parts, as

$$\hat{\rho}(t) = Q^{-1}\exp(-\hat{A}+\hat{B}), \tag{1}$$

$$\text{where, } \hat{A}(t) = \int d^3x\Big[\beta^\nu(\vec{x},t)\hat{T}_{0\nu}(\vec{x},t) - \alpha(\vec{x},t)\hat{N}^0(\vec{x},t) - \frac{1}{2}\Omega_{\alpha\beta}(\vec{x},t)\hat{\Sigma}^{0\alpha\beta}(\vec{x},t)\Big], \tag{2}$$

$$\hat{B}(t) = \int d^3x\int_{-\infty}^{t} dt_1\, e^{\epsilon(t_1-t)}\hat{C}(\vec{x},t_1), \tag{3}$$

$$\hat{C} = \hat{T}_{\mu\nu}\partial^\mu\beta^\nu - \hat{N}^\mu\partial_\mu\alpha - \frac{1}{2}\hat{\Sigma}^{\mu\alpha\beta}\partial_\mu\Omega_{\alpha\beta} - \frac{1}{2}\Omega_{\alpha\beta}\Big(\hat{T}^{\beta\alpha} - \hat{T}^{\alpha\beta}\Big), \tag{4}$$

If we consider the nonequilibrium part $\hat{B}$ as a perturbation, the statistical average of any operator can then be expanded around local equilibrium up to the second order in $\hat{B}$ as [3]:

$$\Big\langle\hat{O}(x)\Big\rangle = \Big\langle\hat{O}(x)\Big\rangle_l + \int d^4x_1\Big(\hat{O}(x),\hat{C}(x_1)\Big) + \int d^4x_1 d^4x_2\Big(\hat{O}(x),\hat{C}(x_1),\hat{C}(x_2)\Big), \tag{5}$$

here, $\int d^4x_1 = \int d^3x_1\int_{-\infty}^{t} dt_1 e^{\epsilon(t_1-t)}$ and the statistical averaging in local equilibrium is denoted by the subscript l. The two-point $\big(\hat{O}(x),\hat{C}(x_1)\big)$ and three-point $\big(\hat{O}(x),\hat{C}(x_1),\hat{C}(x_2)\big)$ correlation functions are defined in [3, 4].

Next, we proceed by writing a tensor decomposition of operators $\hat{T}^{\mu\nu}$, $\hat{N}^\mu$ and $\hat{\Sigma}^{\mu\alpha\beta}$ as

$$\hat{T}^{\mu\nu} = \hat{\varepsilon}u^\mu u^\nu - \hat{p}\Delta^{\mu\nu} + \hat{\pi}^{\mu\nu} + \hat{q}^\mu u^\nu - \hat{q}^\nu u^\mu + \hat{\tau}^{\mu\nu},$$
$$\hat{N}^\mu = \hat{n}u^\mu + \hat{j}^\mu,$$
$$\hat{\Sigma}^{\mu\alpha\beta} = u^\mu\hat{S}^{\alpha\beta},$$

respectively. Here $\hat{p}$ is the actual isotropic pressure. We used the phenomenological form of the spin tensor, and we are working in the Landau frame ($h_\mu = 0$).

All the operators, except the pressure ($\hat{p}$), can be directly matched with the corresponding hydrodynamic quantities by taking the statistical average using full

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nonequilibrium statistical operator, *viz.* energy density $\varepsilon \equiv \langle \hat{\varepsilon} \rangle$, whereas $\langle \hat{p} \rangle \equiv p + \Pi$, with p being the equilibrium pressure. Furthermore, it is important to note that u^μ, ε, p, n, $\Delta_{\mu\nu}$ are zeroth order term $\mathcal{O}(\partial^0)$ in gradient expansion, whereas, the dissipative terms Π, q^μ, j^μ, $\pi^{\mu\nu}$, $\tau^{\mu\nu}$ are first order in gradient expansion $\mathcal{O}(\partial^1)$. In this work, we assign $\Omega_{\alpha\beta}$ as $\mathcal{O}(\partial^1)$ and $S^{\alpha\beta}$ as $\mathcal{O}(\partial^0)$ in terms of hydrodynamic gradient expansion.

Using the decomposition of $\hat{T}_{\mu\nu}$, $\hat{N}_\mu$ and $\hat{\Sigma}_{\mu\alpha\beta}$ in (4), we separate $\hat{C}$ into two parts, containing first-order ($\hat{C}_F$) and second-order ($\hat{C}_S$) gradient terms, respectively. A crucial step while calculating the nonequilibrium part $\hat{C}$ is that we treat spin density ($S^{\alpha\beta}$) as an independent thermodynamic variable apart from the energy density (ε) and particle density (n). Consequently, any thermodynamic quantity such as β, can be written as a function of these variables, i.e., $\beta \equiv \beta(\varepsilon, n, S^{\alpha\beta})$. Finally, the nonequilibrium part ($\hat{C}$) is used to calculate the statistical average of $\hat{\tau}_{\mu\nu}$, $\hat{q}_\mu$, $\hat{\pi}_{\mu\nu}$ and Π up to second order. Furthermore, these equations are used to derive the evolution equation for the corresponding dissipative tensors. For detailed calculation, see [4].

Evolution equations and Discussion

The evolution equations derived for $\tau_{\mu\nu}$ and $\pi_{\mu\nu}$ are given by

$$\begin{aligned}
&\tau_{\mu\nu} + \tau_\tau \dot{\tau}_{\mu\nu} = 2\gamma\xi_{\mu\nu} + \tilde{\gamma}_\tau \tau_{\mu\nu}\theta + 2\gamma_\omega \Gamma \xi_{\mu\nu}\theta \\
&+ \gamma_{\tau S}\mathcal{K}_{\mu\nu}\theta + 2\gamma_{\tau\tau p}\xi_{\mu\nu}\theta + \gamma_{\tau\tau\tau}\xi_{\alpha\lessdot\mu}\xi_{\nu\gtrdot}^{\ \alpha} \\
&+ \gamma_{\tau\pi\pi}\sigma_{\alpha\lessdot\mu}\sigma_{\nu\gtrdot}^{\ \alpha} + 2\gamma_{\tau\pi\tau}\sigma_{\alpha\lessdot\mu}\xi_{\nu\gtrdot}^{\ \alpha} \\
&+ \gamma_{\tau qq}M_{\lessdot\mu}M_{\nu\gtrdot} + \gamma_{\tau jj}\beta^{-1}\nabla_{\lessdot\mu}\alpha\nabla_{\nu\gtrdot}\alpha \\
&+ 2\gamma_{\tau qj}M_{\lessdot\mu}\nabla_{\nu\gtrdot}\alpha, \qquad (6) \\
&\pi_{\mu\nu} + \tau_\pi \dot{\pi}_{\mu\nu} = 2\eta\sigma_{\mu\nu} + \tilde{\eta}_\tau \theta\pi_{\mu\nu} + 2\eta_\omega \Gamma\sigma_{\mu\nu}\theta \\
&+ 2\eta_{\pi\pi p}\sigma_{\mu\nu}\theta + \eta_{\pi\pi\pi}\sigma_{\alpha<\mu}\sigma_{\nu>}^{\ \alpha} + \eta_{\pi\tau\tau}\xi_{\alpha<\mu}\xi_{\nu>}^{\ \alpha} \\
&+ \eta_{\pi jj}\beta^{-1}\nabla_{<\mu}\alpha\nabla_{\nu>}\alpha + 2\eta_{\pi qj}M_{<\mu}\nabla_{\nu>}\alpha \\
&+ 2\eta_{\pi\pi\tau}\sigma_{\alpha<\mu}\xi_{\nu>}^{\ \alpha} + \eta_{\pi qq}M_{<\mu}M_{\nu>}, \qquad (7)
\end{aligned}$$

respectively. We find that the structure of the evolution equations for the shear stress tensor and the rotational stress tensor are similar, except for an extra term, $\gamma_{\tau S}\mathcal{K}_{\mu\nu}\theta$, which appears in the evolution equation of the rotational stress tensor. This term arises from corrections due to the extended thermodynamic forces via the two-point correlation function. Since we consider the spin density as an independent thermodynamic variable, which is an antisymmetric tensor of rank two, it generates the possibility of a two-point correlation in the presence of the rotational stress tensor, something that is absent in the case of the shear stress tensor. We have also obtained the evolution of bulk viscous pressure given by

$$\begin{aligned}
&\Pi + \tau_\Pi \dot{\Pi} = -\zeta\theta + \tilde{\zeta}_\tau \Pi\theta - \zeta_{S\tau}\mathcal{K}_{\mu\nu}\xi^{\mu\nu} \\
&+ \left[\frac{1}{2}\zeta_{\varepsilon p}^2 \frac{\partial^2 p}{\partial\varepsilon^2} + \frac{1}{2}\zeta_{np}^2 \frac{\partial^2 p}{\partial n^2} + \zeta_{\varepsilon p}\zeta_{np}\frac{\partial^2 p}{\partial n \partial\varepsilon}\right]\theta^2 \\
&- \theta^2[\Gamma\zeta_{pp}^\omega + \tilde{\Gamma}\zeta_{p\varepsilon}^\omega + \tilde{\delta}\zeta_{pn}^\omega] \\
&+ \sum_{i=1,2}\zeta_{p\mathfrak{D}_i}\left[(\partial_{\varepsilon n}^i \beta)\mathcal{X} + (\partial_{\varepsilon n}^i \alpha)\mathcal{Y} + (\partial_{\varepsilon n}^i \Omega_{\mu\nu})\mathcal{Z}^{\mu\nu}\right] \\
&+ \zeta_{ppp}\theta^2 + \zeta_{pqq}M_\mu M^\mu + \zeta_{p\pi\pi}\sigma_{\mu\nu}\sigma^{\mu\nu} \\
&+ \zeta_{p\tau\tau}\xi_{\mu\nu}\xi^{\mu\nu} + \beta^{-1}\zeta_{pjj}\nabla_\mu\alpha\nabla^\mu\alpha. \qquad (8)
\end{aligned}$$

Here, we identify a term, $\zeta_{S\tau}\mathcal{K}_{\mu\nu}\xi^{\mu\nu}$, involving the transport coefficient, given by the correlation of spin density with the rotational stress tensor, in the evolution of bulk viscous pressure. The origin of this term can be traced to the fact that the system now includes an additional independent thermodynamic variable, namely the spin density. Consequently, the equilibrium pressure depends on the spin density. Therefore, any deviation from equilibrium, which is nothing but the bulk viscous pressure, must also depend on the spin density. Finally, the evolution equation of the boost heat vector is obtained as

$$\begin{aligned}
&q_\mu + \tau_q \dot{q}_\mu = -\lambda M_\mu + \tilde{\lambda}_\tau \theta q_\mu + \lambda_\omega \Gamma M_\mu \theta - \lambda\mathcal{Q}_\mu \\
&+ \lambda_{qqp}M_\mu\theta + \lambda_{qq\tau}\xi_{\mu\nu}M^\nu + \lambda_{qq\pi}\sigma_{\mu\nu}M^\nu \\
&+ \lambda_{qj\tau}\xi_{\mu\nu}\nabla^\nu\alpha + \lambda_{qj\pi}\sigma_{\mu\nu}\nabla^\nu\alpha. \qquad (9)
\end{aligned}$$

All the transport coefficients and the other symbols defined in the evolution equations of $\tau_{\mu\nu}$, $\pi_{\mu\nu}$, Π and q_μ are discussed in [4].

Further, we find our results for the $\pi_{\mu\nu}$ and Π at $\mu = 0$ and $\omega_{\alpha\beta} = 0$ matches with the results obtained for nonconformal fluids in flat space-time in [5]. However, our result for Π shows the appearance of an additional term. Detailed discussions are given in [4].

References

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