Mass spectra of heavy quarkonia using Cornell plus harmonic potential

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Introduction

The solution of the non-relativistic radial Schrödinger equation (SE) with spherical symmetrical potentials plays an important role in atomic and hadronic spectroscopy. The non-relativistic quark model [1] is commonly employed for knowing the behavior of heavy hadron. The non-relativistic approximation is good for obtaining the mass spectra of heavy mesons consisiting of heavy quark and antiquark ($\Upsilon(b\bar{b})$, $\psi(c\bar{c})$). This approximation provides a good description of static properties of heavy mesons such as mass spectra, radios etc. while for dynamical properties such as decay, the relativistic corrections are considered. The interaction potential for these system is of Cornell type i.e. Coulomb plus linear terms. The Coulomb term is to be liable for the interaction at small distances and linear term leads to the confinement. This type of interaction potential is accompanimented by lattice quantum charomodynamics calculations [2]. The quark-antiquark interaction has also been studied using Coulomb plus power potential[3].
Here in the present work, we ruminate the confine potential for the quark-antiquark interaction potential which abides Cornell plus harmonic potential terms as

$$U(r) = ar^2 + br - \frac{c}{r}, \quad a > 0. \qquad (1)$$

Although the problem of obtaining the actual behavior of the interquark potential is still a mystry but its solution is essential for detecting the mass spectra for coupled states and for characterzing the electromagnetic characteristics of mesons.

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The SE for two particles interacting via a spherically symmetric potential (1) can be written as

$$\Big[\frac{d^2}{d\,r^2} - \frac{l(l+1)}{r^2} + 2\mu(E - ar^2 - br + \frac{c}{r})\Big] R(r) = 0. \quad (2)$$

where l, μ and E denote the angular momentum quantum number, reduced mass of the particles (for charmonium $\mu = \frac{m_c}{2}$, bottomonium $\mu = \frac{m_b}{2}$) and energy eigenvalues respectively. For accessing the approximate solution of equation (2), a plaussible choice of the wavefunction is made as:

$$R(r) = r^{l+1} exp(-\alpha r^2 - \beta r) g(r). \qquad (3)$$

On substituting the equation (3) into equation (2), we obtain

$$rg^{''}(r) + \Big[-4\alpha r^2 - 2\beta r + 2(l+1)\Big]g^{'}(r) + \Big[(4\alpha^2 - 2\mu a)r^3 + (4\alpha\beta - 2\mu b)r^2 + (\beta^2 - 2(2l+3)\alpha + 2\mu E)r + 2\mu c - 2(l+1)\beta\Big]g(r) = 0, \quad (4)$$

which suggests the values of $\alpha = \sqrt{\frac{a\mu}{2}}$ and $\beta = b\sqrt{\frac{\mu}{2a}}$. Using values of α and β, the equation (4) becomes

$$rg^{''}(r) + \Big[-2\sqrt{2a\mu}\, r^2 - b\sqrt{\frac{2\mu}{a}}\, r + 2(l+1)\Big] g^{'}(r) + \Big[(\frac{b^2}{2a} - (2l+3)\sqrt{2a\mu} + 2\mu E)r + 2\mu c - 2(l+1)b\sqrt{\frac{\mu}{2a}}\Big]g(r) = 0. \quad (5)$$

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In order to obtain the polynomial solutions $g(r) = \sum_{k=0}^{n} a_k r^k$ of this equation, we use the following theorem [4] that characterizes the polynomial solutions of a class of differential equations given by

$$(a_{3,0}x^3 + a_{3,1}x^2 + a_{3,2}x + a_{3,3})y'' + (a_{2,0}x^2 + a_{2,1}x + a_{2,2})y' - (\tau_{1,0}x + \tau_{1,1})y = 0. \quad (6)$$

where $a_{3,i}, i = 0,1,2,3, a_{2,j}, j = 0,1,2$ and $\tau_{1,k}, k = 0,1$ are arbitrary constant parameters, yet to be determined.
Theorem 1. The second order linear differential equation (6) has a polynomial solution of degree n if

$$\tau_{1,0} = n(n-1)a_{3,0} + na_{2,0}, \; n = 0,1,2.... \quad (7)$$

Here, $\tau_{1,0}$ is fixed by equation (7) for a given value of n: the degree of the polynomial solution. Thus the polynomial solution of equation (6), using equation (7), we obtain

$$E_{nl} = \sqrt{\frac{a}{2\mu}}(2n + 2l + 3) - \frac{b^2}{4a}. \quad (8)$$

The condition on the potential parameters are determined as

$$c = (n + l + 1)b\sqrt{\frac{1}{2a\mu}} \quad (9)$$

For determining the mass spectra of the heavy quarkonium systems such as charmonium and bottomonium consist of quark and antiquark of same flavor, we use the following relation [3]

$$M = 2m + E_{nl}. \quad (10)$$

Thus, using equation (8) into (10), the mass spectra of charmonium becomes

$$M_c = 2m_c + \sqrt{\frac{a}{m_c}}(2n + 2l + 3) - \frac{b^2}{4a}. \quad (11)$$

For mass spectra of bottomonium, we change M_c, m_c by M_b, m_b respectively in equation (11). With the help of Matlab programming, the given problem is solved numerically and the results are listed in Tables (i) and (ii).

TABLE I:
Mass spectra of charmonium
($m_c = 1.48GeV, a = 0.042GeV^3, b = 0.255GeV^2$) in GeV

States	c	Numerical	Eq.(11)	Exp. [5]
1S	1.023	3.078	3.078	3.068
1P	2.046	3.415	3.415	3.525
2S	2.046	3.787	3.415	3.663
1D	3.069	3.752	3.752	3.770
2P	3.069	4.030	3.752	–
3S	3.069	4.028	3.752	4.159
4S	4.092	4.280	4.089	4.421

TABLE II:
Mass spectra of bottomonium
($m_b = 4.68GeV, a = 0.143GeV^3, b = 0.465GeV^2$) in GeV

States	c	Numerical	Eq.(11)	Exp. [5]
1S	0.57	9.506	9.464	9.460
1P	1.14	9.841	9.862	9.900
2S	1.14	9.772	9.862	10.023
1D	1.71	10.196	10.214	10.161
2P	1.71	10.468	10.214	10.260
3S	1.71	10.488	10.214	10.355
4S	2.28	10.747	10.566	10.580

Conclusions

The mass spectra of quarkonia potential has been computed both analytically and numerically and the obtained results are in good agreement with the experimental results.

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