# On lower bound of relaxation time for massless fluid in presence of magnetic field

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Present work has gone through analytic calculation of shear viscosity and entropy density of massless relativistic fluid, facing external magnetic field $B$ and then explored its fluid property.

Before going to finite magnetic field picture, let us remind the expression of shear viscosity for bosonic/fermionic fluid in absence of magnetic field [1],

$$\eta = \frac{\beta}{15}\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{\boldsymbol{p}^4}{\omega^2}\tau_c f(1-af)\ , \qquad (1)$$

where $f = 1/(e^{\beta\omega}+a)$, with $a = \pm 1$ for fermionic/bosonic fluid, $\omega = \sqrt{\boldsymbol{p}^2+m^2}$ is energy of massive constituent of medium, and most important quantity is the relaxation time $\tau_c$, which proportionally control the strength of shear viscosity.

Now, when we come the magnetic field picture, an additional time scale $\tau_B = \omega/(eB)$ will come into the picture and in terms of these two time scale $\tau_c$ and $\tau_B$, we will get an effective time scale

$$\tau_{\rm eff} = \tau_c/\left\{1+\left(\frac{\tau_c}{\tau_B}\right)^2\right\} \qquad (2)$$

which will be entered in place of $\tau_c$ of Eq. (1) for finite magnetic field picture. Hence we will get shear viscosity of bosonic/fermionic fluid in presence of magnetic field [2] as

$$\eta_2 = \frac{\beta}{15}\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{\boldsymbol{p}^4}{\omega^2}\tau_{\rm eff} f(1-af)\ , \qquad (3)$$

where the subscript 2 stands for one of the component of shear viscosities among 5 components, appeared due to magnetic field.

For massless case, Eq. (1) becomes

$$\begin{aligned}\eta &= \frac{4\,\tau_c}{5\pi^2}\zeta(4)\,T^4 = \frac{4\pi^2\,\tau_c}{450}\,T^4 \quad \text{for Boson}\\ &= \left(\frac{7}{8}\right)\frac{4\,\tau_c}{5\pi^2}\zeta(4)\,T^4 = \frac{7\pi^2\,\tau_c}{900}\,T^4 \quad \text{for Fermion}\end{aligned} \qquad (4)$$

and Eq. (3) becomes

$$\begin{aligned}\eta_2 &= \frac{\eta(B=0)}{1+(\tau_c/\tau_B)^2} = \frac{\frac{4\pi^2\,\tau_c}{450}\,T^4}{1+(\tau_c/\tau_B)^2} \quad \text{for Boson}\\ &= \frac{\eta(B=0)}{1+(\tau_c/\tau_B)^2} = \frac{\frac{7\pi^2\,\tau_c}{900}\,T^4}{1+(\tau_c/\tau_B)^2} \quad \text{for Fermion}\ .\end{aligned} \qquad (5)$$

For calculation simplification, we have considered average energy in $\tau_B$

$$\begin{aligned}\tau_B &= \left\{\frac{\zeta(4)}{\zeta(3)}\right\}\frac{3T}{eB} \quad \text{for Boson}\\ &= \left\{\frac{7\zeta(4)}{2\zeta(3)}\right\}\frac{3T}{eB} \quad \text{for Fermion}\end{aligned} \qquad (6)$$

Now, let us come to the perfect fluid aspects of fermionic/bosonic fluid. Fluidity of the medium is measured by the shear viscosity to entropy density ratio $\eta/s$, where entropy density $s$ of medium can be expressed as

$$s = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\left(\omega + \frac{\boldsymbol{p}^2}{3\omega}\right) f\ , \qquad (7)$$

whose massless limit is

$$\begin{aligned}s &= \frac{4}{\pi^2}\zeta(4)\,T^3 = \frac{4\pi^2}{90}\,T^3 \quad \text{for Boson}\\ &= \left(\frac{7}{8}\right)\frac{4}{\pi^2}\zeta(4)\,T^3 = \frac{7\pi^2}{180}\,T^3 \quad \text{for Fermion}\ .\end{aligned} \qquad (8)$$

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In classical case, we can imagine a perfect fluid, having $\eta/s = 0$ but in quantum case, we get a lower bound of $\eta/s$, which is also known as KSS bound [3], and it is $1/(4\pi)$. For massless medium, using $\eta$ from Eq. (4) and $s$ from Eq. (8), we can get $\eta/s = \tau_c T/5$, which is interestingly same for bosonic or fermionic fluid, although their individual $\eta$ and $s$ expressions are different. If at $\tau_c = \tau_c^0$, lower bound of $\eta/s$ for massless medium is achieved, then we get

$$\frac{\eta}{s} = \frac{\tau_c^0 T}{5} = \frac{1}{4\pi}$$
$$\Rightarrow \tau_c^0 = \frac{5}{4\pi T} , \qquad (9)$$

which is quite standard well-known results. In this context, the present article provide similar type of analytic expression of relaxation time as a function $T$ and $B$ for massless fluid in presence of magnetic field. Using normal viscosity component $\eta_2$ from Eq. (5) and $s$ from Eq. (8), we will get $\eta_2/s$, which will be now different for Maxwell-Boltzmann (MB), Bose-Einstein (BE) and Fermi-Dirac (FD) distribution cases as it contains $\tau_B$, whose average values are different for different cases. By restricting $\eta_2/s = 1/4\pi$, we can get quadratic equation of $\tau_c$:

$$\tau_c^2 - 4\pi\tau_B^2\frac{\eta(B=0)}{s} + \tau_B^2 = 0$$
$$\Rightarrow \tau_c^2 - \Big(\frac{4\pi T\tau_B^2}{5}\Big)\tau_c + \tau_B^2 = 0$$
$$\Rightarrow \tau_c^2 - \Big(\frac{\tau_B^2}{\tau_c^0}\Big)\tau_c + \tau_B^2 = 0 . \qquad (10)$$

The solution of above equation is

$$\tau_c = \tau_c^{2\pm} = \frac{\tau_B^2}{2\tau_c^0}\Big[1 \pm \sqrt{1 - 4\Big(\frac{\tau_c^0}{\tau_B}\Big)^2}\Big] . \qquad (11)$$

So far from our best knowledge, we are first time addressing an *analytic* expressions of $\tau_c(T, B)$ [2], where massless bosonic/fermionic matter in presence of magnetic field reach the KSS bound. To get a physical solution of Eq. (11), we need

$$1 - 4\Big(\frac{\tau_c^0}{\tau_B}\Big)^2 \geq 0$$
$$\Rightarrow \tau_B \geq 2\tau_c^0 . \qquad (12)$$

Using MB relation $\tau_B = \frac{3T}{eB}$ in above inequality, we have

$$\frac{3T}{eB} \geq 2\frac{5}{4\pi T}$$
$$T \geq \Big(\frac{5eB}{6\pi}\Big)^{1/2} . \qquad (13)$$

Corresponding FD and BE relations from Eq. (6) will give

$$T \geq \Big[\Big(\frac{\zeta(3)}{\zeta(4)}\Big)\frac{5eB}{6\pi}\Big]^{1/2} \text{ for BE}$$
$$T \geq \Big[\Big(\frac{2\zeta(3)}{7\zeta(4)}\Big)\frac{5eB}{6\pi}\Big]^{1/2} \text{ for FD} . \qquad (14)$$

Hence, Eqs. (13) and (14) can be identified as upper allowed domain, where KSS bound can be achieved while lower domain is forbidden zone if we believe that $\eta_2/s$ never goes below $\frac{1}{4\pi}$. Within the allowed $T$-$B$ zone, we will get two positive values of relaxation time $\tau_c^{\pm}$, where KSS bound can be achieved. It can be understandable mathematically as follows. For smaller values of $\tau_c$ or $\tau_c/\tau_B << 1$, $\eta_2/s \propto \tau_c$, while for larger values of $\tau_c$ or $\tau_c/\tau_B >> 1$, $\eta_2/s \propto 1/\tau_c$. Due to this two opposite trends of $\tau_c$ dependence, we are getting a lower and upper values of $\tau_c$, where $\eta_2/s$ reach the lower bound $1/(4\pi)$. The detail analysis of this investigation is documented in Ref. [2].

## References

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