Landau–Lifshitz pseudotensor in the presence of cosmological constant

M. Vaseem Akram[1]*, Anver Aziz[2]
[1,2]Department of Physics, Jamia Millia Islamia, New Delhi-110025, INDIA
* Email: moh158947@st.jmi.ac.in

Abstract

We try to introduce cosmological constant in Landau-Lifshitz energy-momentum pseudotensor with the help of a term which is proportional to metric tensor and distance.

Introduction

When the Landau–Lifshitz pseudotensor was formulated, it was commonly assumed that the cosmological constant was zero [1]. In late 1990 Cosmological constant was discovered to be nonzero, suggesting the fact that our universe is currently expanding [2]. In gravitational field, the four momenta of matter alone must not be conserved, rather the four momenta of matter plus gravitational field. In the theory of general relativity, a stress–energy–momentum pseudotensor, such as the Landau–Lifshitz pseudotensor, is an extension of the non-gravitational stress–energy tensor that incorporates the energy–momentum of gravity. It allows the energy–momentum of a system of gravitating matter to be defined. It allows the total of matter plus the gravitating energy–momentum to form a conserved current within the framework of general relativity, so that the total energy–momentum crossing the hypersurface (3-dimensional boundary) of any compact space–time hypervolume (4-dimensional submanifold) vanishes.

The approach applied here to calculate Landau-Lifshitz pseudotensor is similar as used by 'Classical Theory of Fields' by Landau-Lifshitz [1].

Calculations

To determine the total four momentum of matter plus gravitational field we proceed as follows. We start with Einstein's equation with consmological constant,

$$R^{\text{ik}} - \frac{1}{2} g^{\text{ik}} + \Lambda g^{\text{ik}} = \frac{8\pi G}{c^4} T^{\text{ik}} \quad (1)$$

It can be easily seen with the help of standard Einstein's equation that the covarient derivative of left side of above equation is zero.

$$\left(R^{\text{ik}} - \frac{1}{2} g^{\text{ik}} + \Lambda g^{\text{ik}}\right)_{;} = 0 \quad (2)$$

which let's us conclude that $T^{\text{ik}}{}_{,k} = 0$. Now, we choose a system of coordinates of such form that at some particular point in space-time all the first derivatives of g_{ik} vanish. At such points, the covarient derivative is reduced into partial derivative. If $T^{\text{ik}}{}_{,k} = 0$ then one can conclude that T^{ik} has the form,

$$T^{\text{ik}} = \frac{\partial \eta^{\text{ikl}}}{\partial x^l} \quad (3)$$

where η^{ikl} is antisymetric in k and l,

$$\eta^{\text{ikl}} = -\eta^{\text{ilk}} \quad (4)$$

To bring T^{ik} in this form, we start from equation (1),

$$T^{\text{ik}} = \frac{c^4}{8\pi G}\left(R^{\text{ik}} - \frac{1}{2} g^{\text{ik}} R + \Lambda g^{\text{ik}}\right) \quad (5)$$

In this local system of coordinates, we can approximate the first derivative of Γ to be,

$$\Gamma^{i}{}_{\text{jk},m} = \frac{1}{2}\left(g^{\text{il}} g_{\text{jl,km}} + g^{\text{il}} g_{\text{kl,jm}} - g^{\text{il}} g_{\text{jk,lm}}\right) \quad (6)$$

From which Riemann tensor can be calculated which comes out to be,

$$R^{i}{}_{\text{jkl}} = -\Gamma^{i}{}_{\text{jk},l} + \Gamma^{i}{}_{\text{jl},k} = \frac{g^{\text{im}}}{2}\left(g_{\text{lm,jk}} - g_{\text{jl,mk}} - g_{\text{km,jl}} + g_{\text{jk,ml}}\right) \quad (7)$$

By contracting one index of Riemann tensor, we get the Ricci curvature tensor,

$$R^{\text{ik}} = \frac{1}{2} g^{\text{im}} g^{\text{kp}} g^{\text{ln}}\left(g_{\text{lp,mn}} + g_{\text{mn,lp}} - g_{\text{ln,mp}} - g_{\text{mp,ln}}\right) \quad (8)$$

By contracting, we get the Ricci scalar,

$$R = \frac{1}{2} g^{\text{im}} g^{\text{kp}} g^{\text{ln}}\left(g_{\text{lp,mn}} + g_{\text{mn,lp}} - g_{\text{ln,mp}} - g_{\text{mp,ln}}\right) \quad (9)$$

To introduce Λ , we introduce a new term,

$$\zeta^{ikl} = \frac{2\Lambda}{3}\left(g^{ik}x^l - g^{il}x^k\right) \quad (10)$$

which has the property of anti-symmetry in its two indices. Also, it can be verified that

$$\zeta^{ikl} = -\zeta^{ilk}$$

Using the identity $g_{nk,lm} = -g_{ni}g_{kj}g^{ij}{}_{,lm}$ and equation (8), (9) and (11), after some manipulation, we bring equation (1) in the form,

$$T^{ik} = \frac{\partial}{\partial x^l}\left(\frac{c^4}{16\pi G}\frac{1}{(-g)}\frac{\partial}{\partial x^m}\left((-g)\left(g^{ik}g^{lm} - g^{il}g^{km}\right)\right) + \frac{c^4}{16\pi G}\frac{2\Lambda}{3}\frac{1}{(-g)}\left((-g)\left(g^{ik}x^l - g^{il}x^k\right)\right)\right)$$

the quantity in square bracket is η^{ikl} in the form as required by equation (3). In a local space-time, first derivative of g_{ik} vanish, so we can bring (-g) out of the derivative sign. The above equation now takes the form,

$$(-g)T^{ik} \cong \frac{\partial}{\partial x^l}\left(\frac{c^4}{16\pi G}\frac{\partial}{\partial x^m}\left((-g)\left(g^{ik}g^{lm} - g^{il}g^{km}\right)\right) + \frac{c^4}{16\pi G}\frac{2\Lambda}{3}\left((-g)\left(g^{ik}x^l - g^{il}x^k\right)\right)\right)$$

This relation is derived under the assumption $g_{ik,l} = 0$ which is no longer valid when we go to an arbitrary system of coordinates. In the general case, $\frac{\partial}{\partial x^l}h^{ikl} - (-g)T^{ik}$ is different from zero; we denote it by $(-g)t^{ik}$. The quantity t^{ik} is symmetric in i and k.

$$\frac{\partial}{\partial x^l}h^{ikl} = (-g)\left(T^{ik} + t^{ik}\right)$$

Using the exact expression of T^{ik} and h^{ikl} from equation (12) and (13) respectively, we get after a rather lengthy calculation the final expression for pseudotensor to be,

$$t^{ik}(\Lambda) = \frac{c^4}{16\pi G}\Big[\big(2\Gamma^b{}_{b\beta}\Gamma^\beta{}_{ml} - \Gamma^\beta{}_{\beta l}\Gamma^c{}_{cm} - \Gamma^c{}_{lu}\Gamma^u{}_{mc}\big)\left(g^{il}g^{km} - g^{ik}g^{lm}\right) + g^{il}g^{mn}\big(\Gamma^k{}_{lu}\Gamma^u{}_{mn} + \Gamma^j{}_{jl}\Gamma^k{}_{mn} - \Gamma^k{}_{ln}\Gamma^c{}_{cn} - \Gamma^d{}_{nl}\Gamma^k{}_{md}\big) + g^{kl}g^{mn}\big(\Gamma^i{}_{lj}\Gamma^j{}_{mn} + \Gamma^t{}_{tl}\Gamma^i{}_{mn} - \Gamma^m{}_{mj}\Gamma^i{}_{ld} - \Gamma^i{}_{ln}\Gamma^l{}_{lm}\big) + g^{lm}g^{np}\big(\Gamma^i{}_{ln}\Gamma^k{}_{mp} - \Gamma^i{}_{lm}\Gamma^k{}_{np}\big) + \frac{4\Lambda}{3}g^{ik}x^l\Gamma^\beta{}_{\beta l} - \frac{2\Lambda}{3}g^{i\beta}x^l\Gamma^k{}_{\beta l} - \frac{2\Lambda}{3}g^{kj}x^l\Gamma^i{}_{jl} - \frac{2\Lambda}{3}g^{il}x^k\Gamma^\beta{}_{\beta l} + \frac{2\Lambda}{3}g^{lj}x^k\Gamma^i{}_{lj}\Big]$$

which can be rewritten as

$$t^{ik}(\Lambda) = t^{ik} + \frac{c^4\Lambda}{16\pi G}\left[\frac{4}{3}g^{ik}x^l\Gamma^\beta{}_{\beta l} - \frac{2}{3}g^{i\beta}x^l\Gamma^k{}_{\beta l} - \frac{2}{3}g^{kj}x^l\Gamma^i{}_{jl} - \frac{2}{3}g^{il}x^k\Gamma^\beta{}_{\beta l} + \frac{2}{3}g^{lj}x^k\Gamma^i{}_{lj}\right]$$

Where t^{ik} is the Landau-Lifshitz pseduotensor as obtained by Landau-Lifshitz in the absense of Cosmological constant. It is clear from above expression that this expression goes to zero when Λ =0.

Result and Discussion

In calculating expression of $t^{ik}(\Lambda)$we get all the terms of t^{ik} as derived by Landau-Lifshitz[1]. We also get some extra terms which are explicitly Λ dependent. These extra terms vanish as Λ goes to zero. The extra terms are also space-time dependent, which shows that the absolute value of $t^{ik}(\Lambda)$ is not physical.

References

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