Equilibrium Structure of Magnetized White Dwarf

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Introduction

White dwarfs are massive enough comparable to the mass of the sun, their radii are of the order of earth radius. Therefore, the density of white dwarfs is enormously high which makes them important and there has been a series of investigations to find out the reason behind their mechanical equilibrium, stability under various cosmological conditions, evolution, merger events etc. Chandrasekhar in his pioneering work considered the non-relativistic degenerate electron gas as to be responsible for the star's stability against gravitational pull.

Later the Chandrasekhar mass limit has been modified [1] after considering the relativistic electron gas to be interacting with the nuclei. Recent observations have discovered $\sim$ 250 white dwarfs with well determined magnetic fields [2]. The magnitude of the magnetic field has been further confirmed by other surveys like Sloan digital sky survey, hamburg quasar survey. The method developed in ref-[1] has been taken forward to develop equation of state for the magnetized white dwarf matter [3]. It is worth mentioning the fact that, in all these works spherical equation of equilibrium i.e, Tolman-Oppenheimer-Volkoff (TOV) equation has been implemented. As magnetic field induces pressure anisotropy in the system, an alternative is set to be sought to treat the mechanical equilibrium of magnetized white dwarf.

Calculations and Results

The motive of this work is to develop equations of mechanical equilibrium of a magnetized white dwarf where pressure anisotropy is present due to the magnetic field. In the context of Newtonian relativity and within the framework of Hartle formalism, equilibrium equations for rotating white dwarfs are already developed in ref-[4] where the effect of magnetic anisotropy on the structure of white dwarf is ignored. In the present work, the effect of magnetic field pressure anisotropy is treated similarly using Hartle co-ordinate within the context of Newtonian relativity.

Magnetic Field: In the rest frame of the white dwarf, the magnetic field is directed along z axis. Therefore, the total energy ρ, pressure perpendicular to magnetic field $P_\perp$ and pressure parallel to magnetic field $P_\parallel$ at a point inside the star yields to be,

$$\rho = \rho_m + \frac{B^2}{8\pi} \tag{1}$$

$$P_\perp = P_m + \frac{B^2}{8\pi} \tag{2}$$

$$P_\parallel = P_m - \frac{B^2}{8\pi}, \tag{3}$$

where, ρ_m and P_m are the energy density and pressure due to the white dwarf matter, respectively and $\frac{B^2}{8\pi}$ is the contribution from magnetic field. In general, the total pressure P can be written as,

$$P = P_m + \frac{B^2}{8\pi}\left(1 - 2\cos^2\theta\right) \tag{4}$$

so that the $\theta = 0$ corresponds to $P_\parallel$ and $\theta = \pi/2$ corresponds to $P_\perp$. A density dependent magnetic field profile [3] has been used for the present calculation.

Equations of Equilibrium: The equations are developed using Hartle co-ordinate, where the magnetic field is treated as perturbation $\left(\frac{B^2}{8\pi} \ll P_m\right)$. The unperturbed profile of the spherical white dwarf in the absence of magnetic field is described by R, Θ

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co-ordinates. A new set of co-ordinates, to describe the perturbed profile in presence of magnetic field, has been defined as,

$$r = R + \xi \ , \ \xi \ll R \quad (5)$$
$$\theta = \Theta \quad , \quad (6)$$

which are known as Hartle co-ordinates. A spherical shell of constant density in an unperturbed star deforms in presence of the magnetic field keeping the density same as the unperturbed spherical shell and a point on the spherical shell (R, Θ) displaces to (r, θ) in the deformed configuration. As the star is symmetric with respect to the equatorial plane in presence of magnetic field, the perturbation term ξ can keep only even powers in B. However, in this work we keep upto B^2 term in ξ.

Corresponding to these, the density profile and the consequent equation of state for the magnetized configuration can be written as,

$$\rho_m(r, \theta) = \rho_m(R, \Theta) = \rho_m(R) = \rho_m^{(0)}(R),$$
$$P_m(\rho) = P_m(R, \Theta) = P_m(R) = P_m^{(0)}(R) \quad (7)$$

where, $\rho_m^{(0)}(R)$ and $P_m^{(0)}(R)$ are the corresponding quantities in the unperturbed configuration. The energy density $\rho_B\,(r, \theta)$ and pressure $P_B\,(r, \theta)$ due to magnetic field are dependent on $\rho_m^{(0)}(R)$. Finally, the equation of the hydrostatic equilibrium turns out to be,

$$\Rightarrow \int \frac{d\,(P_m(r, \theta) + P_B(r, \theta))}{\rho(r, \theta)} + \Phi(r, \theta) = \text{Const.} \quad (8)$$

In the deformed configuration, the equation of gravitational potential $\Phi(r, \theta)$ becomes,

$$\nabla^2 \Phi(r, \theta) = 4\pi G \rho(r, \theta), \quad (9)$$

The equations of equilibrium for the magnetized deformed configuration can easily be transformed in R, Θ coordinate after decomposing ξ and Φ as,

$$\xi(R, \Theta) = \sum_l \xi_l(R) P_l(\cos\Theta),$$
$$\Phi(R, \Theta) = \Phi^{(0)}(R) + \Phi^{(2)}(R, \Theta) + O(B^4),$$
$$\Phi^{(2)}(R, \Theta) = \sum_l \Phi_l^{(2)}(R) P_l(\cos\Theta). \quad (10)$$

$P_l(\cos\Theta)$ are legendre polynomials and $\Phi^{(0)}(R)$ is the potential in the unperturbed configuration, whereas, the $\Phi^{(2)}(R, \Theta)$ term represents the perturbation $\sim B^2$ in magnetized configuration. Thence, one can find the equilibrium equations to be,

$$\int \frac{dP_m^{(0)}(R)}{\rho_m^0(R)} + \Phi^{(0)}(R) = \text{Const} \ (\ B^0 \ );$$
$$\frac{1}{3} \int \frac{dP_B^{(0)}(R)}{\rho_m^0(R)} + \xi_0(R) \frac{d\Phi^{(0)}}{dR} + \Phi_0^{(2)}(R)$$
$$+ \int \frac{\rho_B^0(R)}{\rho_m^0(R)} d\Phi^{(0)}(R) = 0, \ (B^2 \ \ \& \ \ l = 0);$$
$$-\frac{4}{3} \int \frac{dP_B^{(0)}(R)}{\rho_m^0(R)} + \xi_2(R) \frac{d\Phi^{(0)}}{dR} + \Phi_2^{(2)}(R) = 0,$$
$$(B^2 \ \ \& \ \ l = 2); \quad (11)$$

and the potential equations to be,

$$\nabla^2 \Phi^{(0)}(R) = 4\pi G \rho_m^{(0)}(R), \quad (\ B^0 \ );$$
$$\xi_0(R) \frac{d}{dR} \nabla^2 \Phi^{(0)}(R) + \nabla^2 \Phi_0^{(2)}(R) = 4\pi G \rho_B^{(0)}(R),$$
$$(B^2 \ \ \& \ \ l = 0);$$
$$\xi_2(R) \frac{d}{dR} \nabla^2 \Phi^{(0)} + \nabla^2 \Phi_2^{(2)}(R) - \frac{6}{R^2} \Phi_2^{(2)}(R) = 0,$$
$$(B^2 \ \ \& \ \ l = 2); \quad (12)$$

These equations are valid only under the approximation that, the pressure due to magnetic field at any point inside the star is much smaller than the matter pressure P_m and corresponding numerical calculations are under progress.

References

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