# Energy eigenvalue spectra of Radial Schrodinger Equation for Instanton and Sextic potential by the Power Series Method


Sanhita Sengupta[1], * Praveen P Dsouza [1,2]

[1] *Department of Physics, St Joseph's University, Bangalore -560027, Karnataka, INDIA*

[2] *Department of Physics, St Aloysius (Deemed to be University), Mangalore-575003, Karnataka, INDIA*


## Introduction

The power series approach is used to find the solutions to the three-dimensional Schrödinger equation for the Instanton and Sextic (IS) potentials, taking into consideration the significance of these potentials. The particle interaction controlled by IS potential is understood through the use of approximations and numerical techniques. A potential energy function that has the shape of a degree six polynomial is known as a sextic potential. To compute the eigenvalues and eigenfunctions of a three-dimensional radial Schrodinger Equation for IS potential, our methodology involves choosing a suitable ansatz for the radial wave function. By using this technique, we will establish a connection between the IS potential's energy eigenvalues and the Hulthen plus Linear Potential(HLP), and then we will investigate how both potentials can be used to produce the mass-spectra of heavy quarkonium. The proposed approach is appropriate both for analytic calculations and for numerical computations. This method allows also to determine the spectrum of the Schrödinger equation with different potential.

## Theoretical Background

The heavy quark potential derived from the instanton ensemble rises linearly as the relative distance between the quark and anti quark increases and gets saturated. As the quark and the antiquark distance increases the central potential turns out to be Coulomb like potential. Hence to study the mass spectra of the quarkonia we have added Coulombic type potential to central instanton potential. This can be understood as a non perturbative contribution to the perturbative potential from instanton vacuum at large inter quark distances greater than the instanton size. [1]. The potential that we are using here is:

$$V_c(r) = \frac{4\pi\rho^3}{R^4 N_c}\left(1.345\frac{r^2}{\rho^2} - 0.501\frac{r^4}{\rho^4}\right) \qquad (1)$$

Where for mesons the number of colors $N_c$=3 and the average separation between instantons is R = 1fm. And $\rho$ is the the average size of the instanton.

$$Let,\ \frac{4\pi\rho}{3}(1.345) = k_1 \qquad (2)$$

$$while,\ \frac{4\pi}{3\rho}(0.501) = k_2 \qquad (3)$$

$$V_c(r) = k_1 r^2 + (-k_2) r^4 \qquad (4)$$

$$V_s(r) = k_3 r^6 \qquad (5)$$

$$V(r) = V_c(r) + V_s(r) \qquad (6)$$

Final form of the radial Schrödinger equation is

$$\frac{-\hbar^2}{2m}\left(\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right) + \frac{-l(l+1)R(r)}{r^2}\right) + V(r)R(r) = E'R(r)$$

The N-dimensional radial Schrödinger equation for the potential V(r) is written as

$$\frac{d^2R(r)}{dr^2} + \frac{(N-1)}{r}\frac{d}{dr} + \frac{2\mu}{\hbar^2}\left[E' - V(r) - \frac{l(l+N-2)\hbar^2}{2\mu r^2}\right] R(r)=0$$

$$\left[\frac{d^2}{dr^2} + \frac{(N-1)}{r}\frac{d}{dr} - \frac{l(l+N-2)\hbar^2}{2\mu r^2} + (E - a_3 r^2 + b_3 r^2 - c_3 r^6)\right]R(r)=0 \qquad (7)$$

Where

$E = \frac{2\mu E'}{\hbar^2}$ $\quad a_3 = \frac{2\mu k_1}{\hbar^2}$ $\quad b_3 = \frac{2\mu k_2}{\hbar^2}$ $\quad c_3 = \frac{2\mu k_3}{\hbar^2}$

Using an suitable ansatz for the radial wavefunction

$$R(r) = e^{-\delta r^2+\beta r^4}F(r)$$

Where $\alpha$ and $\beta$ are positive constants. And F(r) given by,

$$F(r) = \sum_{n=0}^{\infty} d_n r^{n+l} \quad (8)$$

Substituting equation (8) in equation (7) and on equating the coefficients of various powers of r to zero, we obtain the following relations.

$$(n+l)(n+l+N-2) - l(l+N-2) = 0$$
$$a_3 - 4\beta(2n+2l+N+2) - 4\delta^2 = 0$$
$$E - 2\delta(2n+2l+N) = 0$$
$$-16\delta\beta + b_3 = 0$$
$$16\beta^2 - c_3 = 0$$

The solution of above set of equations gives energy eigen values and eigen function

$$E' = \frac{k_2}{2}\sqrt{\frac{\hbar^2}{2\mu k_3}}(2n+2l+N)$$

$\delta = \dfrac{\mu k_2\sqrt{2}}{4\hbar^2\sqrt{\frac{\mu k_3}{\hbar^2}}}$ and $\beta = \frac{1}{4}\sqrt{\frac{2\mu k_3}{\hbar^2}}$

The relation between $k_1, k_2$ and $k_3$ is given by

$$k_1 = \sqrt{\frac{\hbar^2 k_3}{2\mu}}(2n+2l+N+2) + \frac{k_2^2}{4k_3}$$

## Results and discussion:

In the present work, we have analyticaly obtained the bound state solutions to the non-relativistic Schrondinger Equation for the IS potential using power series method. The lowest energy eigenvalues of the IS potential in three dimensions with different values of n and l quantum numbers were computed in table 1. For $\hbar = \mu = 1$ and N =3 is the dimension.

| n | l | $\rho$ | $k_1 \; 10^{-2}$ | $k_2$ | $k_3 . \; 10^{-3}$ | $E'$ |
|---|---|---|---|---|---|---|
| 1 | 0 | 0.33 | 0.94 | -6.29 | 0.13 | -4.94 |
| 2 | 0 | 0.33 | 0.94 | -6.29 | 0.13 | -6.92 |
| 3 | 0 | 0.33 | 0.94 | -6.29 | 0.13 | -8.90 |
| 1 | 1 | 0.33 | 0.94 | -6.29 | 0.14 | -6.66 |
| 2 | 1 | 0.33 | 0.94 | -6.29 | 0.14 | -8.57 |
| 3 | 1 | 0.33 | 0.94 | -6.29 | 0.14 | -10.4 |
| 1 | 2 | 0.33 | 0.94 | -6.29 | 0.15 | -8.90 |
| 2 | 2 | 0.33 | 0.94 | -6.29 | 0.15 | -10.8 |
| 3 | 2 | 0.33 | 0.94 | -6.29 | 0.15 | -12.8 |

**Table 1**: The lowest Energy eigen values of IS potential

The energy eigenvalues of the IS potential can be used to compute the mass spectra of five mesons. Since eigenstate solutions are achieved via power series method for N dimensional wave equation in an easy way, the power series method can be handled as an efficient method for exact solutions of other higher dimensional wave equations for different type potentials.

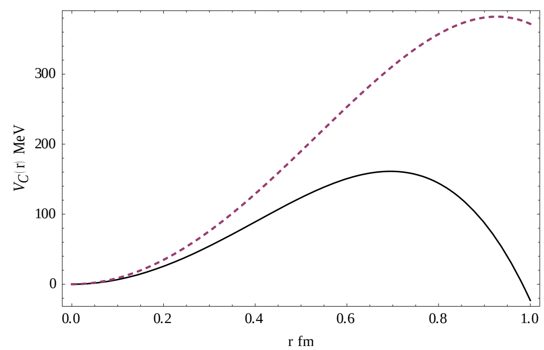


**Fig 1**: Graphical Representation of $V_c(r)$ verses r

As an application of the linear combination potentials IS and HLP can calculate the mass spectra of heavy quarkonium using the equation

$$M = 2m + E'$$

We expect that the solution obtained here will have fruitful applications in areas where IS and Hulthen Plus linear potential models are relevant such as in particle physics and high energy physics

## References


[1] Communications in Theoretical Physics, Volume 71, Number 2 DOI 10.1088/0253-6102/71/2/192

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[3] Energy eigenvalue spectra and applications of the sextic and the Coulomb perturbed potentials. Phys. Scr. 97 (2022) 055301